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THE BI-GRADED STRUCTURE OF SYMMETRIC ALGEBRAS WITH APPLICATIONS TO REES RINGS

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ABSTRACT. Consider a rational projective plane curve $\mathcal{C}$ parameterized by three homogeneous forms h_1, h_2, h_3 of the same degree d in the polynomial ring $R = k[x,y]$ over the field k. Extracting a common factor, we may harmlessly assume that the ideal $I = (h_1, h_2, h_3)R$ has height two. Let φ be a homogeneous minimal Hilbert-Burch matrix for the row vector $[h_1, h_2, h_3]$. So, φ is a 3×2 matrix of homogeneous forms from R; the entries in column m have degree d_m, with $d_1 \le d_2$ and $d_1 + d_2 = d$. The Rees algebra $\mathcal{R}$ of I is the subring $k[h_1t, h_2t, h_3t]$ of the polynomial ring $k[t]$. The bi-projective spectrum of $\mathcal{R}$ is the graph of the parameterization of $\mathcal{C}$; and therefore, there is a dictionary which translates between the singularities of $\mathcal{C}$ and the algebra structure of $\mathcal{R}$. The ring $\mathcal{R}$ is the quotient of the symmetric algebra $\mathrm{Sym}(I)$ by the ideal $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I))$ of local cohomology with support in the homogeneous maximal ideal $\mathfrak{m} = (x,y)$ of R. We denote this local cohomolgy module by $\mathcal{A}$; it is the ideal of $\mathrm{Sym}(I)$ which defines $\mathcal{R}$. The ideal $\mathcal{A}_{\ge d_2-1}$, which is an approximation of $\mathcal{A}$, can be calculated using linkage. We exploit the bi-graded structure of $\mathrm{Sym}(I)$ in order to describe the structure of an improved approximation $\mathcal{A}_{\ge d_1-1}$ when $d_1 < d_2$ and φ has a generalized zero in its first column. (The later condition is equivalent to assuming that $\mathcal{C}$ has a singularity of multiplicity d_2.) In particular, we give the bi-degrees of a minimal bi-homogeneous generating set for this ideal. When $2 = d_1 < d_2$ and φ has a generalized zero in its first column, then we record explicit generators for $\mathcal{A}$. When $d_1 = d_2$, we provide a translation between the bi-degrees of a bi-homogeneous minimal generating set for $\mathcal{A}_{d_1-2}$ and the number of singularities of multiplicity d_1 which are on or infinitely near $\mathcal{C}$. We conclude with a table which translates between the bi-degrees of a bi-homogeneous minimal generating set for $\mathcal{A}$ and the configuration of singularities of $\mathcal{C}$ in the case that the curve $\mathcal{C}$ has degree six.

1. INTRODUCTION.

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Our basic setting is as follows: Let k be an algebraically closed field, $R = k[x,y]$ a polynomial ring in two variables, and I an ideal of R minimally generated by homogeneous forms h_1, h_2, h_3 of the same degree $d > 0$. Extracting a common divisor we may harmlessly assume that I has height two. We will keep these assumptions throughout the introduction, though many of our results are stated and proved in greater generality.

On the one hand, the homogeneous forms h_1, h_2, h_3 define a morphism

$$\eta : \mathbb{P}^1_k \xrightarrow{[h_1:h_2:h_3]} \mathbb{P}^2_k \tag{1.0.1}$$

whose image is a curve $\mathcal{C}$. After reparametrizing we may assume that the map η is birational onto its image or, equivalently, that the curve $\mathcal{C}$ has degree d.

On the other hand, associated to h_1, h_2, h_3 is a syzygy matrix φ that gives rise to a homogeneous free resolution of the ideal I,

$$0 \longrightarrow R(-d-d_1) \oplus R(-d-d_2) \xrightarrow{\varphi} R(-d)^3 \longrightarrow I \longrightarrow 0 \; .$$

Here φ is a 3 by 2 matrix with homogeneous entries in R, of degree d_1 in the first column and of degree d_2 in the second column. We may assume that $d_1 \le d_2$. Notice that $d = d_1 + d_2$ by the Hilbert-Burch Theorem.

The two aspects, the curve $\mathcal{C}$ parametrized by the forms h_1, h_2, h_3 and the syzygy matrix φ of these forms, are mediated by the Rees algebra $\mathcal{R}$ of I. The Rees algebra is defined as the subalgebra $R[It] = R[h_1t, h_2t, h_3t]$ of the polynomial ring $R[t]$. It becomes a standard bi-graded k-algebra if one sets deg $x =$ deg $y = (1,0)$ and deg $t = (-d,1)$, which gives deg $f_it = (0,1)$. The bi-projective spectrum of $\mathcal{R}$ is the graph $\Gamma \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$ of the morphism $\eta = [h_1 : h_2 : h_3]$. Projecting to the second factor of $\mathbb{P}^1_k \times \mathbb{P}^2_k$ one obtains a surjection $\Gamma \twoheadrightarrow \mathcal{C}$, which corresponds to an inclusion of coordinate rings

$$\mathcal{R} \hookleftarrow k[h_1t, h_2t, h_3t] \; .$$

Thus the coordinate ring $A(\mathcal{C})$ of the curve $\mathcal{C}$ can be recovered as a direct summand of the Rees algebra $\mathcal{R}$, namely

$$A(\mathcal{C}) = \bigoplus_i \mathcal{R}_{(0,i)} \; .$$

The same holds for the ideal I,

$$I \simeq It = \bigoplus_i \mathcal{R}_{(i,1)} \; .$$

Finally, the inclusion $\Gamma \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$ corresponds to a homogeneous epimorphism $\mathcal{R} \twoheadleftarrow B$, where $B = k[x,y,T_1,T_2,T_3]$ is a bi-graded polynomial ring with deg $x =$ deg $y = (1,0)$ and deg $T_i = (0,1)$, and the variables T_i are mapped to h_it. The kernel of this epimorphism is a bi-homogeneous ideal $\mathcal{J}$

of B, the 'defining ideal' of the Rees algebra $\mathcal{R}$. Now the syzygy module of I can be recovered as well,

$$\operatorname{syz}(I) \simeq \bigoplus_i \mathcal{J}_{(i,1)} \ .$$

Thus, the philosophy underlying this work can be summarized as follows: One wishes to study local properties of the rational plane curve $\mathcal{C}$, such as the types of its singularities, by means of the syzygies of I, since linear relations among polynomials are easier to handle than polynomial relations. The mediator is the Rees algebra, which in turn carries more information than the coordinate ring $A(\mathcal{C})$ of the curve, just like the graph of a map reveals more than the image of the map. One may therefore hope that even relatively simple numerical data associated to this algebra, such as the (first) bi-graded Betti numbers, say a great deal about the curve. The syzygies of I appear in the defining ideal $\mathcal{J}$, which leads one to study defining ideals of Rees algebras. Finding such ideals or, equivalently, describing Rees rings explicitly in terms of generators and relations, is a fundamental problem in elimination theory, that has occupied commutative algebraists, algebraic geometers, and, more recently, applied mathematicians. The problem is wide open, even for ideals of polynomial rings in two variables.

Write $\mathfrak{m} = (x,y)$ for the homogeneous maximal ideal of $R = k[x,y]$ and S for the polynomial ring $k[T_1,T_2,T_3]$. Recall that $B = k[x,y,T_1,T_2,T_3] = R \otimes_k S$ and that $\mathcal{R} = B/\mathcal{J}$. To study the Rees algebra of an ideal one customarily maps the symmetric algebra onto it,

$$0 \longrightarrow \mathcal{A} \longrightarrow \operatorname{Sym}(I) \longrightarrow \mathcal{R} \longrightarrow 0 \ .$$

One readily sees that $\mathcal{A} = \mathrm{H}^0_{\mathfrak{m}}(\operatorname{Sym}(I)) = 0 :_{\operatorname{Sym}(I)} \mathfrak{m}^\infty$. A presentation of the symmetric algebra is well understood, $\operatorname{Sym}(I) \simeq B/(g_1,g_2)$, where

$$[g_1,g_2] = [T_1,T_2,T_3] \cdot \varphi \ .$$

The polynomials g_i are homogeneous of bi-degree $(d_i,1)$ and together they form a B-regular sequence. The symmetric algebra $\operatorname{Sym}(I)$, the Rees algebra $\mathcal{R}$, and the ideal $\mathcal{A}$ of $\operatorname{Sym}(I)$ which defines $\mathcal{R}$ all are naturally equipped with two gradings: the T-grading and the (x,y)-grading. Both gradings play crucial roles in our work. The T-grading is often used in the study of symmetric algebras and Rees algebras. For example, an ideal is said to have "linear type" if all of the defining equations of the Rees algebra have T-degree 1. Ideals of linear type are much studied in the literature; see, for example [13, 18, 11, 17]. Much of our work is focused on the xy-grading. We view $\mathcal{A}$ as $\bigoplus \mathcal{A}_i$, where $\mathcal{A}_i$ is the S-submodule of $\mathcal{A}$ which consists of all elements homogeneous in x and y of degree i; in other words, $\mathcal{A}_i = \bigoplus_j \mathcal{A}_{(i,j)}$. One major advantage of this decomposition is the fact that $\mathcal{A}_i$ is non-zero for only a finitely many values of i. Both gradings come into to play in the proof of Theorem 3.3, which is one of the main results of the paper. In Theorem 3.3 we identify the degrees of the minimal generators of each $\mathcal{A}_i$. In particular, for each fixed i we must determine the minimum value of j for which $\mathcal{A}_{i,j}$ is not zero. Curiously enough, the key point in our proof is that we consider a graded strand of $\mathcal{A}$ with j-fixed and i varying.

The mathematics that sets the present project in motion is due to Jouanolou [21, 20]; see also Busé [4]. Jouanolou proved that the multiplication map

$$\mathcal{A}_i \otimes \operatorname{Sym}(I)_{\delta-i} \longrightarrow \mathcal{A}_\delta \simeq S(-2) \tag{1.0.2}$$

gives a perfect pairing of S-modules for $\delta = d-2$. Jouanolou uses Morley forms to exhibit dual bases for the modules of (1.0.2). The perfect pairing (1.0.2) shows that the S-module structure of $\mathcal{A}_i$ is completely determined by the S-module structure of $\operatorname{Sym}(I)_{\delta-i}$. The symmetric algebra $\operatorname{Sym}(I)$ is a complete intersection defined by the regular sequence g_1, g_2; so, the S-module structure of $\operatorname{Sym}(I)_{\delta-i}$ depends on the relationship between $\delta - i$, d_1, and d_2.

Ultimately we offer three proofs of Jouanolou's perfect pairing (1.0.2). Two of our arguments are different than Jouanolou's; furthermore, our arguments are self-contained, and we obtain results not obtained by Jouanolou. In particular, we relate the entries of φ to module-theoretic properties of $\mathcal{A}_i$ and $\operatorname{Sym}(I)_{\delta-i}$, (see especially Theorem 2.11 and Corollary 2.12 in Subsection 2.B) and also to information about the singularities of the curve parameterized by a minimal generating set for I; see especially Sections 6 and 7. A very quick proof of the abstract duality relating $\mathcal{A}$ and $\operatorname{Sym}(I)$ is given in Subsection 2.A. This proof computes the local cohomology with support in $\mathfrak{m}$ along the Koszul complex which resolves $\operatorname{Sym}(I)$ as a B-module. In Subsection 2.C we take advantage of the module theoretic properties of the $\mathcal{A}_i$ (in particular the fact that they are reflexive S-modules as was shown in Subsection 2.B) to prove that the abstract isomorphism of Subsection 2.A is actually given by multiplication. Finally, in Theorem 4.2, as part of our review of the theory of Morley forms in Section 4, we give Jouanolou's own proof of the perfect pairing (1.0.2).

In Theorem 3.3 we describe the S-module structure of $\mathcal{A}_{\geq d_1-1}$ (according to the convention described above, $\mathcal{A}_{\geq d_1-1}$ means $\bigoplus_j A_{(\geq d_1-1,j)}$) under the hypothesis that $d_1 < d_2$ and φ has a generalized zero in its first column. This module is free and we identify the bi-degrees of a bi-homogeneous basis for it; see also Table 3.5. In Corollary 3.10 we identify the bi-degrees of a minimal bi-homogeneous generating set of $\mathcal{A}_{\geq d_1-1}$ as an ideal of $\operatorname{Sym}(I)$. When one views this result in the geometric context of (1.0.1), then the hypothesis concerning the existence of a generalized zero is equivalent to assuming that $\mathcal{C}$ has a singularity of multiplicity d_2, and the ideal $\mathcal{A}_{\geq d_1-1}$ of the conclusion is an approximation of the ideal $\mathcal{A}$ which defines the graph $\Gamma \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$ of the parameterization $\eta : \mathbb{P}^1_k \twoheadrightarrow \mathcal{C}$. The part of $\operatorname{Sym}(I)$ that corresponds to $\mathcal{A}_{\geq d_1-1}$, under the duality of (1.0.2), is $\operatorname{Sym}(I)_{\leq d_2-1}$. There is no contribution from g_2 to the S-module $\operatorname{Sym}(I)_{\leq d_2-1}$ in the bi-homogeneous B-resolution of $\operatorname{Sym}(I)$. So, basically, we may ignore g_2. Furthermore, the hypothesis that the first column of φ has a generalized zero allows us to make the critical calculation over a subring U of S, where U is a polynomial ring in two variables.

In Section 5 we focus on the situation $2 = d_1 < d_2$. Busé [4] has given explicit formula for the generators of $\mathcal{A}$ if the first column of φ does not have a generalized zero. In Theorem 5.11 we carry out the analogous project in the case that the first column of φ does have a generalized zero. These hypotheses about generalized zeros in the first column have geometric implications for the corresponding curve. In Busé's case all of the singularities of $\mathcal{C}$ have multiplicity at most

d_1; whereas, in the situation of Theorem 5.11, $\mathcal{C}$ has at least one singularity of multiplicity d_2. The proof of Theorem 5.11 is based on the results of Section 3 (since $\mathcal{A}_{\geq d_1-1}$ is equal to $\mathcal{A}_{\geq 1}$ when $d_1 = 2$ and $\mathcal{A}_0$ is always well understood), an analysis of the kernel of a Toeplitz matrix of linear forms in two variables (see Lemma's 5.7 and 5.10), and Jouanolou's theory of Morley forms. We review the theory of Morley forms in Section 4.

In Section 6 we completely describe the S-module structure of $\mathcal{A}_{d_1-2}$ when $d_1 = d_2$. A preliminary version of this section initiated the investigation that culminated in [6]. The geometric significance of these calculations are emphasized in [6] and are reprised in the present paper; however the main focus of Section 6 is on the Rees algebras.

The results in Sections 1 – 6 suffice to provide significant information about the defining equations for $\mathcal{R}$ if $d = d_1 + d_2 \leq 6$, since then $d_1 \leq 2$ (see, especially, Section 5) or $d_1 = d_2$ (see, especially, Section 6). Section 7 is concerned with the case $d = 6$, the case of a sextic curve. We show that there is, essentially, a one-to-one correspondence between the bi-degrees of the defining equations of $\mathcal{R}$ on the one hand and the types of the singularities on or infinitely near the curve $\mathcal{C}$ on the other hand.

If R is a ring, then we write $\operatorname{Quot}(R)$ for the *total quotient ring* of R; that is, $\operatorname{Quot}(R) = U^{-1}R$, where U is the set of non zerodivisors on R. If R is a domain, then the total quotient ring of R is usually called the *quotient field* of R.

If M is a matrix, then M^{T} denotes the transpose of M. If M has entries in a k-algebra, where k is a field, then a *generalized zero* of M is a product $pMq^{\mathrm{T}} = 0$, where p and q are non-zero row vectors with entries from k.

If M is a $\ell - 1 \times \ell$ matrix with entries in a ring then the ring elements $m_1, \dots, m_\ell$ are the *signed maximal minors* of the matrix M if m_i is $(-1)^{i+1}$ times the determinant of the submatrix of M that is obtained by removing column i. If $m_1, \dots, m_\ell$ are the signed maximal minors of the matrix M, then the product $M[m_1, \dots, m_\ell]^{\mathrm{T}}$ is zero.

If S is a ring and A, B, and C are S-modules, then the S-module homomorphism $F : A \otimes_S B \to C$ is a *perfect pairing* if the induced S-module homomorphisms $A \to \operatorname{Hom}_S(B,C)$ and $B \to \operatorname{Hom}_S(A,C)$, which are given by $a \mapsto F(a \otimes \_)$ and $b \mapsto F(\_ \otimes b)$, are isomorphisms.

2. Duality, perfect pairing, and consequences.

Data 2.1. Let k be a field, $R = k[x,y]$ a polynomial ring in 2 variables over k, $\mathfrak{m} = (x,y)R$ the homogeneous maximal ideal of R, and I a height 2 ideal of R minimally generated by 3 forms of the same positive degree d. Let $\delta = d - 2$. Let φ be a homogeneous Hilbert-Burch matrix for I; each entry in column i of φ has degree d_i with $d_1 \leq d_2$. Let $\mathcal{A}$ be the kernel of the natural surjection

$$\operatorname{Sym}(I) \twoheadrightarrow \mathcal{R}$$

from the symmetric algebra of I to the Rees algebra of I, and let S and B be the polynomial rings $S = k[T_1, T_2, T_3]$ and $B = R \otimes_k S = k[x, y, T_1, T_2, T_3]$. View B as a bi-graded k-algebra, where x and y have bi-degree $(1,0)$ and each T_i has bi-degree $(0,1)$. A presentation of the symmetric algebra is

given by $\operatorname{Sym}(I) \simeq B/(g_1, g_2)$, where

$$[g_1, g_2] = [T_1, T_2, T_3] \cdot \varphi \ .$$

Remarks 2.2. Adopt Data 2.1. The Hilbert-Burch Theorem guarantees that $d_1 + d_2 = d$. One readily sees that the $\operatorname{Sym}(I)$-ideals

$$\mathcal{A} \quad \text{and} \quad \mathrm{H}^0_{\mathfrak{m}}(\operatorname{Sym}(I)) = 0 :_{\operatorname{Sym}(I)} \mathfrak{m}^\infty$$

are equal. The polynomial g_m is homogeneous of bi-degree $(d_m, 1)$. The polynomials g_1, g_2 form a regular sequence on B because the dimension of $\operatorname{Sym}(I)$ is equal to 3 by [19]. Thus, the Koszul complex provides a bi-homogeneous B-resolution of the symmetric algebra:

$$K_\bullet(g_1, g_2; B) \longrightarrow \operatorname{Sym}(I) \to 0 \ . \tag{2.2.1}$$

Remark 2.3. When the bi-graded B-modules $\mathcal{A} = \bigoplus_{i,j} \mathcal{A}_{(i,j)}$ and $\operatorname{Sym}(I) = \bigoplus_{i,j} \operatorname{Sym}(I)_{(i,j)}$ are viewed as S-modules, then we write $\mathcal{A} = \bigoplus_i \mathcal{A}_i$ and $\operatorname{Sym}(I) = \bigoplus_i \operatorname{Sym}(I)_i$, where $\mathcal{A}_i$ represents the S-module $\mathcal{A}_i = \bigoplus_j \mathcal{A}_{(i,j)}$ and $\operatorname{Sym}(I)_i$ represents the S-module $\operatorname{Sym}(I)_i = \bigoplus_j \operatorname{Sym}(I)_{(i,j)}$.

The goal of this section is to prove that $\mathcal{A}_\delta$ is a free S-module generated by an explicit element $\mathrm{syl} \in \operatorname{Sym}(I)_{(\delta,2)}$ and that the multiplication map

$$\mathcal{A}_i \otimes \operatorname{Sym}(I)_{\delta-i} \longrightarrow \mathcal{A}_\delta = S \cdot \mathrm{syl} \tag{2.3.1}$$

gives a perfect pairing of S-modules. Both of these results are due to Jouanolou [21, 20]; see also Busé [4]. Our arguments are different from Jouanolou's; furthermore, our arguments are self-contained, and we obtain results not obtained by Jouanolou. In particular, we relate the entries of φ to module-theoretic properties of $\mathcal{A}_i$ and $\operatorname{Sym}(I)_{\delta-i}$, and also to information about the singularities of the curve parameterized by a minimal generating set for I. The section consists of four subsections:

2.A The abstract duality relating $\mathcal{A}$ and $\operatorname{Sym}(I)$;
2.B The torsionfreeness and reflexivity of the S-module $\operatorname{Sym}(I)_i$ and how these properties are related to the geometry of the corresponding curve;
2.C The duality is given by multiplication; and
2.D Explicit S-module generators for $\mathcal{A}_i$, when i is large.

2.A THE ABSTRACT DUALITY RELATING $\mathcal{A}$ AND $\operatorname{Sym}(I)$.

The goal of this subsection is to relate the S-modules $\mathcal{A}_i$ and $\operatorname{Sym}(I)_{\delta-i}$ and to express $\mathcal{A}_i$ as the kernel of a homomorphism of free S-modules. This goal is attained in Corollary 2.5 and Theorem 2.7. The first step toward (2.3.1) is to establish an abstract isomorphism between $\mathcal{A}$ and a shift of $\underline{\operatorname{Hom}}_S(\operatorname{Sym}(I), S)$, where $\underline{\operatorname{Hom}}$ denotes the graded dual. Toward that aim, one computes local cohomology with support in $\mathfrak{m}$ along the resolution (2.2.1), uses the symmetry of the Koszul complex, and the isomorphism $\mathrm{H}^2_{\mathfrak{m}}(R) \simeq \underline{\operatorname{Hom}}_k(R, k)(2)$.

Theorem 2.4. *If Data* 2.1 *is adopted, then there is an isomorphism of bi-graded B-modules*

$$\mathcal{A} \simeq \underline{\mathrm{Hom}}_S(\mathrm{Sym}(I), S)(-\delta, -2).$$

Proof. We first establish two isomorphisms that are essential for our proof. From the self-duality of the Koszul complex one obtains an isomorphism of complexes of bi-graded B-modules,

$$\underline{\mathrm{Hom}}_S(K_\bullet(g_1, g_2; B), S) \simeq K_\bullet(g_1, g_2; \underline{\mathrm{Hom}}_S(B, S))[2](d, 2). \tag{2.4.1}$$

The symbol [2] indicates homological degree shift. The internal bi-degree shift is written $(d, 2)$. We also use the following isomorphisms of bi-graded B-modules,

$$\begin{array}{rcll} \mathrm{H}^2_{\mathfrak{m}}(B) & \simeq & \mathrm{H}^2_{\mathfrak{m}}(R \otimes_R B) & \\ & \simeq & \mathrm{H}^2_{\mathfrak{m}}(R) \otimes_R B & \text{since } B \text{ is } R\text{-flat} \\ & \simeq & \mathrm{H}^2_{\mathfrak{m}}(R) \otimes_k S & \text{since } B \simeq R \otimes_k S \\ & \simeq & \underline{\mathrm{Hom}}_k(R, k)(2) \otimes_k S & \text{by Serre duality} \\ & \simeq & \underline{\mathrm{Hom}}_S(B, S)(2, 0). & \end{array}$$

We deduce that

$$\mathrm{H}^2_{\mathfrak{m}}(B) \simeq \underline{\mathrm{Hom}}_S(B, S)(2, 0). \tag{2.4.2}$$

To prove the assertion of the theorem we decompose $K_\bullet = K_\bullet(g_1, g_2; B)$ into short exact sequences

$$0 \to K_2 \longrightarrow K_1 \longrightarrow \mathcal{J} \to 0 \qquad \text{and} \qquad 0 \to \mathcal{J} \longrightarrow K_0 \longrightarrow \mathrm{Sym}(I) \to 0.$$

The second sequence gives

$$0 = \mathrm{H}^0_{\mathfrak{m}}(K_0) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I)) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(\mathcal{J}) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(K_0) = 0, \tag{2.4.3}$$

where the first and last modules vanish because grade $\mathfrak{m}B > 1$. The first sequence above yields

$$0 = \mathrm{H}^1_{\mathfrak{m}}(K_1) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(\mathcal{J}) \longrightarrow \mathrm{H}^2_{\mathfrak{m}}(K_2) \xrightarrow{\partial} \mathrm{H}^2_{\mathfrak{m}}(K_1). \tag{2.4.4}$$

Notice that ∂ is the second differential of the Koszul complex $K_\bullet(g_1, g_2; \mathrm{H}^2_{\mathfrak{m}}(B))$ because the formation of local cohomology commutes with taking direct sums. Thus from (2.4.3) and (2.4.4) we obtain a bi-graded isomorphism

$$\mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I)) \simeq \mathrm{H}_2(K_\bullet(g_1, g_2; \mathrm{H}^2_{\mathfrak{m}}(B)).$$

On the other hand,

$$\begin{array}{rcll} \mathrm{H}_2(K_\bullet(g_1, g_2; \mathrm{H}^2_{\mathfrak{m}}(B)) & \simeq & \mathrm{H}_2(K_\bullet(g_1, g_2; \underline{\mathrm{Hom}}_S(B, S)(2, 0)) & \text{by (2.4.2)} \\ & \simeq & \mathrm{H}_0(\underline{\mathrm{Hom}}_S(K_\bullet(g_1, g_2, B), S))(-d, -2)(2, 0) & \text{by (2.4.1)} \\ & \simeq & \underline{\mathrm{Hom}}_S(\mathrm{Sym}(I), S)(2 - d, -2). & \end{array}$$

The last isomorphism holds because $K_\bullet(g_1, g_2; B)$ is a resolution of $\mathrm{Sym}(I)$. ☐

Corollary 2.5. *Adopt Data* 2.1. *The following statements hold.*

(1) *The graded S-modules $\mathcal{A}_i$ and* $\mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta - i}, S(-2))$ *are isomorphic for all i.*
(2) *The S-module $\mathcal{A}_i$ is zero for all $i > \delta$.*
(3) *The graded S-module $\mathcal{A}_\delta$ is isomorphic to $S(-2)$.*
(4) *The S-module $\mathcal{A}_i$ is reflexive for all i.*

Proof. Assertion (1) follows directly from Theorem 2.4; (2) follows from (1) since $\operatorname{Sym}(I)_\ell$ is zero when ℓ is negative; (3) holds because $\operatorname{Sym}(I)_0 = S$; and (4) holds because the S-dual of every finitely generated S-module is reflexive. $\square$

Theorem 2.4 shows that the S-module structure of $\mathcal{A}_i$ is completely determined by the S-module structure of $\operatorname{Sym}(I)_{\delta-i}$. The symmetric algebra $\operatorname{Sym}(I)$ is a complete intersection defined by the regular sequence g_1, g_2; so, the S-module structure of $\operatorname{Sym}(I)_{\delta-i}$ depends on the relationship between $\delta - i$, d_1, and d_2. The chart which is given in Theorem 2.7 describes the S-module structure of $\operatorname{Sym}(I)_{\delta-i}$ and $\mathcal{A}_i$ as a function of how $\delta - i$ sits with respect to $d_1 \le d_2$. We set up the relevant notation in the next definition.

Definition 2.6. The polynomials g_1 and g_2 in $S[x,y]$ are defined in Data 2.1. At this point we name their coefficients by writing

$$g_m = \sum_{\ell=0}^{d_m} c_{\ell,m} x^\ell y^{d_m-\ell}, \tag{2.6.1}$$

with $c_{\ell,m} \in S_1$, for m equal to 1 or 2. For positive integers n and m, with m equal to 1 or 2, let $\Upsilon_{n,m}$ be the $(d_m+n)\times n$ matrix

$$\Upsilon_{n,m} = \begin{bmatrix}
c_{0,m} & 0 & 0 & 0 & 0 & 0 \\
c_{1,m} & c_{0,m} & 0 & 0 & 0 & 0 \\
\cdot & c_{1,m} & \ddots & 0 & 0 & 0 \\
\cdot & \cdot & \ddots & \ddots & 0 & 0 \\
\cdot & \cdot & \ddots & \ddots & \ddots & 0 \\
\cdot & \cdot & \ddots & \ddots & \ddots & c_{0,m} \\
c_{d_m,m} & \cdot & \ddots & \ddots & \ddots & c_{1,m} \\
0 & c_{d_m,m} & \ddots & \ddots & \ddots & \cdot \\
0 & 0 & c_{d_m,m} & \ddots & \ddots & \cdot \\
0 & 0 & 0 & \ddots & \ddots & \cdot \\
0 & 0 & 0 & 0 & \ddots & \cdot \\
0 & 0 & 0 & 0 & 0 & c_{d_m,m}
\end{bmatrix},$$

with entries from S_1. The matrix $\Upsilon_{n,m}$ represents the map of free S-modules $S[x,y]_{n-1} \to S[x,y]_{n-1+d_m}$ which is given by multiplication by g_m when the bases $y^{n-1}, \dots, x^{n-1}$ and $y^{n-1+d_m}, \dots, x^{n-1+d_m}$ are used for $S[x,y]_{n-1}$ and $S[x,y]_{n-1+d_m}$, respectively.

Theorem 2.7. *Adopt Data 2.1. The following statements hold.*

(1) *If $0 \le i \le d_1 - 2$, then the S-modules $\mathcal{A}_i$ and $\operatorname{Sym}(I)_{\delta-i}$ both have rank $i+1$; furthermore, the following sequences of S-modules are exact:*

$$0 \to S(-1)^{d_2-i-1} \oplus S(-1)^{d_1-i-1} \xrightarrow{\begin{pmatrix} \Upsilon_{d_2-i-1,1} & \Upsilon_{d_1-i-1,2}\end{pmatrix}} S^{\delta-i+1} \longrightarrow \operatorname{Sym}(I)_{\delta-i} \to 0$$

and

$$0 \to \mathcal{A}_i \to \ S(-2)^{\delta-i+1} \xrightarrow{\begin{pmatrix}\Upsilon_{d_2-i-1,1}{}^{\mathrm{T}}\\ \Upsilon_{d_1-i-1,2}{}^{\mathrm{T}}\end{pmatrix}} S(-1)^{d_2-i-1}\oplus S(-1)^{d_1-i-1}\ .$$

(2) *If* $d_1-1\le i\le d_2-2$, *then the S-modules* $\mathcal{A}_i$ *and* $\mathrm{Sym}(I)_{\delta-i}$ *both have rank* d_1; *furthermore, the following sequences of S-modules are exact*:

$$0\to\ S(-1)^{d_2-i-1} \xrightarrow{\Upsilon_{d_2-i-1,1}} S^{\delta-i+1}\ \longrightarrow \mathrm{Sym}(I)_{\delta-i}\to 0.$$

and

$$0\to\mathcal{A}_i\to\ S(-2)^{\delta-i+1} \xrightarrow{\Upsilon_{d_2-i-1,1}{}^{\mathrm{T}}} S(-1)^{d_2-i-1}\ .$$

(3) *If* $d_2-1\le i\le\delta$, *then the S-modules* $\mathcal{A}_i$ *and* $\mathrm{Sym}(I)_{\delta-i}$ *both have rank* $\delta-i+1$; *furthermore,*

$$\mathrm{Sym}(I)_{\delta-i}\simeq S^{\delta-i+1}\qquad \textit{and}\qquad \mathcal{A}_i\simeq S(-2)^{\delta-i+1}.$$

Proof. The homogeneous B-resolution of $\mathrm{Sym}(I)$

$$0\to B(-d_1-d_2,-2)\longrightarrow B(-d_1,-1)\oplus B(-d_2,-1)\longrightarrow B\longrightarrow \mathrm{Sym}(I)\to 0,$$

which is given in (2.2.1), may be decomposed into the graded strands recorded in the statement of the Theorem. The rank of each S-module $\mathrm{Sym}(I)_{\delta-i}$ can be read immediately from its resolution. The statements about the modules $\mathcal{A}_i$ follow from part (1) of Corollary 2.5. □

2.B THE TORSIONFREENESS AND REFLEXIVITY OF THE S-MODULE $\mathrm{Sym}(I)_i$ AND HOW THESE PROPERTIES ARE RELATED TO THE GEOMETRY OF THE CORRESPONDING CURVE.

We are now going to investigate the torsionfreeness and reflexivity of the graded components of $\mathrm{Sym}(I)$. To do so we need to estimate the height of ideals of minors of the matrices that appear in parts (1) and (2) of Theorem 2.7.

Lemma 2.8. *Adopt Data* 2.1. *Let n be a positive integer and* $\Upsilon_{n,1}$ *be the* $(d_1+n)\times n$ *matrix introduced in Definition 2.6. The following statements hold*:

(1) $\mathrm{ht}\,I_n(\Upsilon_{n,1})\ge 2$;
(2) $\mathrm{ht}\,I_n(\Upsilon_{n,1})=3$ *if and only if the first column of* φ *does not have a generalized zero.*

Proof. The ideal $I_n(\Upsilon_{n,1})$ is equal to the n^{th} power of the ideal $I_1(\Upsilon_{1,1})$; and, for any given ideal J in the polynomial ring S, the ideals J and J^n have the same height. Therefore, it suffices to prove the result when $n=1$. On the other hand, the ideal $I_1(\Upsilon_{1,1})$ is generated by linear forms in S_1; so the height of $I_1(\Upsilon_{1,1})$ is equal to the minimal number of generators of $I_1(\Upsilon_{1,1})$. Recall that

$$[y^{d_1},xy^{d_1-1},\dots,x^{d_1}]\,\Upsilon_{1,1}=g_1=[T_1,T_2,T_3]\,\varphi_1,$$

where φ_1 is the first column of φ. The entries of φ_1 generate an ideal of height at least 2 because $\operatorname{ht} I_2(\varphi) = 2$. To complete the proof it suffices to show that

$$\mu(I_1(\Upsilon_{1,1})) = \mu(I_1(\varphi_1)). \tag{2.8.1}$$

Indeed, suppose, for the time being, that (2.8.1) has been established. Then

$$2 \le \operatorname{ht}(I_1(\varphi_1)) \implies 2 \le \mu(I_1(\varphi_1)) = \mu(I_1(\Upsilon_{1,1})) = \operatorname{ht}(I_1(\Upsilon_{1,1})) = \operatorname{ht}(I_n(\Upsilon_{n,1}))$$

and (1) holds. Also,

$$\operatorname{ht} I_n(\Upsilon_{n,1}) \le 2 \iff \mu(I_1(\Upsilon_{1,1})) \le 2 \iff \mu(I_1(\varphi_1)) \le 2 \iff \varphi_1 \text{ has a generalized zero.}$$

Now we prove (2.8.1). Suppose that $I_1(\Upsilon_{1,1})$ is minimally generated by $\lambda_1, \dots, \lambda_s$ in S_1. It follows that

$$\Upsilon_{1,1} = \lambda_1\rho_1 + \cdots + \lambda_s\rho_s$$

for column vectors ρ_ℓ in $\operatorname{Mat}_{(d_1+1)\times 1}(k)$. For $1 \le \ell \le s$, let ξ_ℓ be the homogeneous form

$$\xi_\ell = [y^{d_1}, xy^{d_1-1}, \dots, x^{d_1}]\rho_\ell$$

in R_{d_1} and let Z_ℓ be the column vector of three constants with $\lambda_\ell = [T_1, T_2, T_3]Z_\ell$. We have

$$\begin{aligned}[T_1,T_2,T_3]\varphi_1 = g_1 &= [y^{d_1}, xy^{d_1-1}, \dots, x^{d_1}]\Upsilon_{1,1} = [y^{d_1}, xy^{d_1-1}, \dots, x^{d_1}](\lambda_1\rho_1 + \cdots + \lambda_s\rho_s)\\ &= \xi_1\lambda_1 + \cdots + \xi_s\lambda_s = \xi_1[T_1,T_2,T_3]Z_1 + \cdots + \xi_s[T_1,T_2,T_3]Z_s = [T_1,T_2,T_3](\sum_{\ell=1}^{s} \xi_\ell Z_\ell).\end{aligned}$$

The entries of the 3×1 vector $\varphi_1 - \sum_\ell \xi_\ell Z_\ell$ are homogeneous forms of degree d_1 in R; hence, this vector cannot be in the kernel of $[T_1, T_2, T_3]$ unless it is already zero. Thus, $\varphi_1 = \sum_\ell \xi_\ell Z_\ell$. Let V be the subspace of R_{d_1} which is spanned by the entries of φ_1. We have shown that V is a subspace of the vector space spanned by $\xi_1, \dots, \xi_s$. It follows that

$$\mu(I_1(\varphi_1)) = \dim V \le s = \mu(I_1(\Upsilon_{1,1})).$$

One may read the calculation in the other direction to see that $\mu(I_1(\Upsilon_{1,1})) \le \mu(I_1(\varphi_1))$. $\square$

Remark 2.9. Adopt Data 2.1 and assume k is algebraically closed. The signed maximal minors h_1, h_2, h_3 of φ define a morphism

$$\mathbb{P}^1_k \xrightarrow{[h_1,h_2,h_3]} \mathbb{P}^2_k$$

whose image is a rational plane curve $\mathcal{C}$. The degree of the curve $\mathcal{C}$ satisfies the equality $\deg \mathcal{C} = d/r$, where r is the degree of the field extension $[\operatorname{Quot}(k[R_d]) : \operatorname{Quot}(k[I_d])]$. In particular, $r = 1$ if and only if the parametrization is birational onto its image.

As it turns out, the heights of various ideals of minors of interest can be expressed in terms of the singularities of the curve $\mathcal{C}$.

Lemma 2.10. *Adopt Data* 2.1 *with* $d_1 = d_2$. *Let* C *be the* $(d_1+1)\times 2$ *matrix* $C = \begin{pmatrix}\Upsilon_{1,1} & \Upsilon_{1,2}\end{pmatrix}$ *for* $\Upsilon_{1,1}$ *and* $\Upsilon_{1,2}$ *as introduced in Definition* 2.6, *and let* $\mathcal{C}$ *be the curve of Remark* 2.9. *The following statements hold*:

(1) $\operatorname{ht} I_2(C) \geq 2$ *if and only if* $I_1(\varphi)$ *is not a complete intersection*; *furthermore, if* k *is algebraically closed, then the previous conditions hold if and only if the curve* $\mathcal{C}$ *is singular*;
(2) $\operatorname{ht} I_2(C) = 3$ *if and only if* φ *does not have a generalized zero*; *furthermore, if* k *is algebraically closed, then the previous conditions hold if and only if the curve* $\mathcal{C}$ *is singular and its singularities have multiplicity at most* $(\deg \mathcal{C})/2-1$.

Proof. We may harmlessly assume that k is algebraically closed. Let r be the degree of the field extension $[\operatorname{Quot}(k[R_d]) : \operatorname{Quot}(k[I_d])]$, as described in Remark 2.9. Then there exists a regular sequence u,v in R_r so that $I = I_2(\varphi')R$ for some 3×2 matrix φ' whose entries are homogeneous polynomials of degree $(\deg \mathcal{C})/2 = d/2r$ in the variables u,v (see [24]). The signed maximal minors of φ' provide a birational parametrization of the same curve $\mathcal{C}$. Let $R' = k[u,v]$ and I' be the ideal $I_2(\varphi')$ of R'. Define elements g'_1, g'_2 in $R'[T_1,T_2,T_3]$ via the equation $[g'_1, g'_2] = [T_1,T_2,T_3]\cdot\varphi'$. Use these data to obtain matrices $\Upsilon'_{n,m}$ as in Definition 2.6. Finally, let C' be the matrix $\begin{pmatrix}\Upsilon'_{1,1} & \Upsilon'_{1,2}\end{pmatrix}$. From [6, 3.14(2)] we know that $\operatorname{ht} I_2(C') \geq 2$ if and only if the curve $\mathcal{C}$ is singular, and $\operatorname{ht} I_2(C') = 3$ if and only if the curve $\mathcal{C}$ is singular and its singularities have multiplicity at most $(\deg \mathcal{C})/2-1$. (The result from [6] is stated assuming the birationality of the parametrization. A complete proof of the geometric interpretation of $\operatorname{ht} I_2(C') \geq 2$ uses the fact that a rational plane curve of degree at least three is singular.)

The curve $\mathcal{C}$ is nonsingular if and only if its homogeneous coordinate ring $k[I'_{d/r}]$ is normal. Since the parametrization is birational, the latter obtains if and only if $k[I'_{d/r}] = k[R'_{d/r}]$ or, equivalently, $3 = \dim_k I'_{d/r} = \dim_k R'_{d/r}$. This holds if and only if $d/r = 2$. The last equality means that $I_1(\varphi')$ is generated by linear forms, equivalently $I_1(\varphi') = (u,v)R'$. The latter holds if and only if $I_1(\varphi')$ is a complete intersection, again because the parametrization is birational. Finally, the R'-ideal $I_1(\varphi')$ is a complete intersection if and only if the R-ideal $I_1(\varphi)$ is.

On the other hand, the curve $\mathcal{C}$ is singular and its singularities have multiplicity at most $(\deg \mathcal{C})/2-1$ if and only if φ' does not have a generalized zero, as was shown in part (4) of [6, 1.9]. Notice that φ' has a generalized zero if and only if φ does.

It remains to show that $I_2(C') = I_2(C)$. Extend the ordered set $v^{d/r}, \ldots, u^{d/r}$ of monomials in u,v of degree d/r to an ordered basis of R_d, which we call $b_0,\ldots,b_d$. Define a $d+1\times 2$ matrix D with entries in S_1 via the equality $[g'_1, g'_2] = [b_0,\ldots,b_d]\cdot D$. Notice that $D = \begin{bmatrix} C' \\ 0 \end{bmatrix}$, and hence $I_2(C') = I_2(D)$. Finally, the matrix D is obtained from C by elementary row operations that correspond to the transition from $b_0,\ldots,b_d$ to the monomial basis $y^d,\ldots,x^d$ of R_d. Therefore $I_2(D) = I_2(C)$. □

Theorem 2.11. *Adopt Data* 2.1 *and let* $\mathcal{C}$ *be the curve of Remark* 2.9. *The following statements hold.*

(1) *If* $d_1 \leq i \leq d_2-1$, *then*
   (a) *the* S*-module* $\operatorname{Sym}(I)_i$ *is torsionfree* ; *and*

(b) *the S-module* $\mathrm{Sym}(I)_i$ *is reflexive if and only if the first column of* φ *does not have a generalized zero*; *furthermore, if k is algebraically closed, then the previous conditions hold if and only if the singularities of the curve $\mathcal{C}$ have multiplicity at most $d_1(\deg \mathcal{C})/d$.*

(2) *If $i = d_1 = d_2$, then*

(a) *the S-module* $\mathrm{Sym}(I)_i$ *is torsionfree if and only if* $I_1(\varphi)$ *is not a complete intersection*; *furthermore, if k is algebraically closed, then the previous conditions hold if and only if the curve $\mathcal{C}$ is singular* ; *and*

(b) *the S-module* $\mathrm{Sym}(I)_i$ *is reflexive if and only if* φ *does not have a generalized zero*; *furthermore, if k is algebraically closed, then the previous conditions hold if and only if the curve $\mathcal{C}$ is singular and its singularities have multiplicity at most* $(\deg \mathcal{C})/2-1$.

Proof. We first argue the third equivalence in item (1.b). Again, as in the proof of Lemma 2.10, one obtains a regular sequence u,v of forms of degree $d/(\deg \mathcal{C})$ so that $I = I_2(\varphi')R$ for some 3×2 matrix φ' whose entries in position i,j are homogeneous polynomials of degree $d_j(\deg \mathcal{C})/d$ in the variables u,v (see [24]). Thus one reduces to the case of a birational parametrization. Now part (4) of [6, 1.9] shows that $\mathcal{C}$ has a singularity of multiplicity at least $d_1(\deg \mathcal{C})/d+1$ if and only if the first column of φ' has a generalized zero, or, equivalently, the first column of φ has a generalized zero.

Write $n = i-d_1+1$. From Theorem 2.7 we know that the S-module $\mathrm{Sym}(I)_i$ has projective dimension at most one and that it is presented by $\Upsilon_{n,1}$ in the setting of (1) and by $C = \begin{pmatrix}\Upsilon_{1,1} & \Upsilon_{1,2}\end{pmatrix}$ in the setting of (2). Thus $\mathrm{Sym}(I)_i$ is torsionfree if and only if $I_n(\Upsilon_{n,1})$ or $I_2(C)$, respectively, has height at least two. Likewise, $\mathrm{Sym}(I)_i$ is reflexive if and only if this height is at least 3. Now it remains to appeal to Lemmas 2.8 and 2.10. $\square$

Corollary 2.12. *Adopt Data* 2.1 *and let $\mathcal{C}$ be the curve of Remark* 2.9. *The following statements hold.*

(1) *If $d_1-1 \le i \le d_2-2$, then the S-module $\mathcal{A}_i$ is free if and only if the first column of the matrix φ has a generalized zero*; *furthermore, if k is algebraically closed, then the previous conditions hold if and only if the curve $\mathcal{C}$ has a singularity of multiplicity equal to $d_2(\deg \mathcal{C})/d$.*

(2) *If $i = d_1 = d_2$, then the S-module $\mathcal{A}_i$ is free if and only if $\mu(I_2(C)) \le 4$*; *furthermore, if k is algebraically closed, then the previous conditions hold if and only if there are at least two singularities of multiplicity* $(\deg \mathcal{C})/2$ *on or infinitely near $\mathcal{C}$.*

Proof. We prove part (1). If the first column of the matrix φ does not have a generalized zero, then $\mathrm{Sym}(I)_{\delta-i}$ is reflexive according to Theorem 2.11 part (1.b). Thus, part (1) of Corollary 2.5 shows that $\mathrm{Sym}(I)_{\delta-i} \cong \mathrm{Hom}_S(\mathcal{A}_i, S(-2))$. Since $\mathrm{Sym}(I)_{\delta-i}$ is not free it follows that $\mathcal{A}_i$ cannot be free either. Conversely, if the first column of the matrix φ has a generalized zero then $\mathcal{A}_i$ is free as will be shown in Theorem 3.3. The third equivalence in item (1) is parts (1), (2), and (4) of [6, Cor. 1.9], after reducing to the case of a birational parameterization.

The first equivalence of part (2) will be proved in Theorem 6.2. The second equivalence follows from [6, 3.22] after reducing to the case of a birational parametrization. $\square$

2.C THE DUALITY IS GIVEN BY MULTIPLICATION.

In (2.3.1), we promised an explicit perfect pairing $\mathcal{A}_i \otimes_S \operatorname{Sym}(I)_{\delta-i} \to A_\delta = S \cdot \text{syl}$. So far, in part (1) of Corollary 2.5, we showed that $\mathcal{A}_i$ is isomorphic to $\operatorname{Hom}_S(\operatorname{Sym}(I)_{\delta-i}, S(-2))$. In Theorem 2.13 we prove that the abstract isomorphism of Corollary 2.5 is given by multiplication. The other highlight of the present subsection is Corollary 2.14, where we prove that the $\operatorname{Sym}(I)$-ideals $\mathcal{A}_{\geq i}$ and $0 :_{\operatorname{Sym}(I)} \mathfrak{m}^{d-1-i}$ are equal. We use this equality in subsection 2.D to record explicit generators for the S-modules $\mathcal{A}_i$ when the equality represents linkage. The Sylvester element syl is one of these explicit generators. It is introduced in Remark 2.17 of subsection 2.D.

Theorem 2.13. *Adopt Data* 2.1. *For each i, the multiplication map* $\mathcal{A}_i \otimes \operatorname{Sym}(I)_{\delta-i} \longrightarrow \mathcal{A}_\delta$ *induces a homogeneous isomorphism of S-modules*

$$\mathcal{A}_i \longrightarrow \operatorname{Hom}_S(\operatorname{Sym}(I)_{\delta-i}, \mathcal{A}_\delta).$$

Proof. If $i < 0$ or $\delta < i$, then the assertion is trivial because $\operatorname{Hom}_S(\operatorname{Sym}(I)_{\delta-i}, \mathcal{A}_\delta)$ and $\mathcal{A}_i$ both vanish due to (1) and (2) from Corollary 2.5. Hence, it suffices to prove the assertion for i in the range $0 \leq i \leq \delta$. We fix such an i and we denote the map induced by multiplication by

$$\Phi : \mathcal{A}_i \longrightarrow \operatorname{Hom}_S(\operatorname{Sym}(I)_{\delta-i}, \mathcal{A}_\delta).$$

Write Σ for $\operatorname{Sym}(I)$. If $\mathfrak{p} \in \operatorname{Spec}(S)$, then the ring $\Sigma_\mathfrak{p} = S_\mathfrak{p} \otimes_S \Sigma$ is a standard graded $S_\mathfrak{p}$-algebra with irrelevant ideal $(\Sigma_\mathfrak{p})_+ = \mathfrak{m}\Sigma_\mathfrak{p}$. Furthermore, $\Sigma_\mathfrak{p}$ is a complete intersection with $\dim \Sigma_\mathfrak{p} = \dim S_\mathfrak{p}$. The source and the target of the homomorphism Φ are reflexive S-modules, see part (4) of Corollary 2.5. Thus, to prove that Φ is injective it suffices to show that $\Phi_\mathfrak{p} = S_\mathfrak{p} \otimes \Phi$ is injective when $\mathfrak{p}$ is the zero ideal of S, and then to prove that Φ is surjective one only needs to check that $\Phi_\mathfrak{p}$ is surjective for every $\mathfrak{p} \in \operatorname{Spec}(S)$ with $\dim S_\mathfrak{p} = 1$.

First, let $\mathfrak{p}$ be the zero ideal. In this case, $\Sigma_\mathfrak{p}$ is an Artinian standard graded Gorenstein algebra over a field with homogeneous maximal ideal $\mathfrak{m}\Sigma_\mathfrak{p}$. Therefore $\mathcal{A}_\mathfrak{p} = 0 :_{\Sigma_\mathfrak{p}} \mathfrak{m}^\infty = \Sigma_\mathfrak{p}$. In particular, $[\Sigma_\mathfrak{p}]_\delta \neq 0$ and $[\Sigma_\mathfrak{p}]_i = 0$ for $i > \delta$ by Corollary 2.5, parts (1) and (2). In other words, $[\Sigma_\mathfrak{p}]_\delta$ is the socle of the Gorenstein algebra $\Sigma_\mathfrak{p}$. Thus, multiplication induces an isomorphism

$$[\Sigma_\mathfrak{p}]_i \longrightarrow \operatorname{Hom}_{S_\mathfrak{p}}([\Sigma_\mathfrak{p}]_{\delta-i}, [\Sigma_\mathfrak{p}]_\delta).$$

As $\Sigma_\mathfrak{p} = \mathcal{A}_\mathfrak{p}$, we conclude that $\Phi_\mathfrak{p}$ is an isomorphism.

Next, let $\mathfrak{p} \in \operatorname{Spec}(S)$ with $\dim S_\mathfrak{p} = 1$. We need to show that $\Phi_\mathfrak{p}$ is surjective. Let θ be any element of $\operatorname{Hom}_{S_\mathfrak{p}}([\Sigma_\mathfrak{p}]_{\delta-i}, [\mathcal{A}_\mathfrak{p}]_\delta)$. We prove that the map θ is multiplication by some element of $[\mathcal{A}_\mathfrak{p}]_i$. Notice that

$$\begin{array}{rcl} \operatorname{Hom}_{S_\mathfrak{p}}([\Sigma_\mathfrak{p}]_{\delta-i}, [\mathcal{A}_\mathfrak{p}]_\delta) & = & \operatorname{Hom}_{S_\mathfrak{p}}((\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p})/(\mathfrak{m}^{\delta-i+1}\Sigma_\mathfrak{p}), [\mathcal{A}_\mathfrak{p}]_\delta) \\ & \subset & \operatorname{Hom}_{\Sigma_\mathfrak{p}}((\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p})/(\mathfrak{m}^{\delta-i+1}\Sigma_\mathfrak{p}), \mathcal{A}_\mathfrak{p}) \\ & \subset & \operatorname{Hom}_{\Sigma_\mathfrak{p}}((\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p})/(\mathfrak{m}^{\delta-i+1}\Sigma_\mathfrak{p}), \Sigma_\mathfrak{p}), \end{array}$$

where the next-to-last inclusion holds because $\mathfrak{m}[\mathcal{A}_\mathfrak{p}]_\delta = 0$ by Corollary 2.5, part (2).

We will prove that $\theta \in [\mathrm{Hom}_{\Sigma_\mathfrak{p}}((\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p})/(\mathfrak{m}^{\delta-i+1}\Sigma_\mathfrak{p}),\Sigma_\mathfrak{p})]_i$ can be lifted to a map

$$\widetilde{\theta} \in \left[\mathrm{Hom}_{\Sigma_\mathfrak{p}}(\Sigma_\mathfrak{p}/(\mathfrak{m}^{\delta-i+1}\Sigma_\mathfrak{p}),\Sigma_\mathfrak{p})\right]_i.$$

Any such $\widetilde{\theta}$ is induced by multiplication by an element $\lambda \in [\Sigma_\mathfrak{p}]_i$. The element λ is necessarily annihilated by $\mathfrak{m}^{\delta-i+1}$. Recall that $\mathcal{A}_\mathfrak{p} = 0 :_{\Sigma_\mathfrak{p}} \mathfrak{m}^\infty$. Thus λ lies in $[\mathcal{A}_\mathfrak{p}]_i$, and therefore $\widetilde{\theta}$ and θ both are induced by multiplication by an element $\lambda \in [\mathcal{A}_\mathfrak{p}]_i$.

To show that θ can be lifted, we first apply $\mathrm{Hom}_{\Sigma_\mathfrak{p}}(-,\Sigma_\mathfrak{p})$ to the short exact sequence of graded $\Sigma_\mathfrak{p}$-modules,

$$0 \to (\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p})/(\mathfrak{m}^{\delta-i+1}\Sigma_\mathfrak{p}) \longrightarrow \Sigma_\mathfrak{p}/(\mathfrak{m}^{\delta-i+1}\Sigma_\mathfrak{p}) \longrightarrow \Sigma_\mathfrak{p}/(\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p}) \to 0.$$

The corresponding long exact sequence of cohomology induces the following exact sequence of $S_\mathfrak{p}$-modules

$$\begin{aligned}\left[\mathrm{Hom}_{\Sigma_\mathfrak{p}}(\Sigma_\mathfrak{p}/(\mathfrak{m}^{\delta-i+1}\Sigma_\mathfrak{p}),\Sigma_\mathfrak{p})\right]_i &\longrightarrow \left[\mathrm{Hom}_{\Sigma_\mathfrak{p}}((\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p})/(\mathfrak{m}^{\delta-i+1}\Sigma_\mathfrak{p}),\Sigma_\mathfrak{p})\right]_i\\ &\longrightarrow \left[\mathrm{Ext}^1_{\Sigma_\mathfrak{p}}(\Sigma_\mathfrak{p}/(\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p}),\Sigma_\mathfrak{p})\right]_i.\end{aligned}$$

It suffices to prove that $[\mathrm{Ext}^1_{\Sigma_\mathfrak{p}}(\Sigma_\mathfrak{p}/(\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p}),\Sigma_\mathfrak{p})]_i = 0$. To do so, recall that $\Sigma_\mathfrak{p} = S_\mathfrak{p}[x,y]/(g_1,g_2)$, where g_1,g_2 is a regular sequence of forms of degrees d_1,d_2. Therefore,

$$\omega_{\Sigma_\mathfrak{p}} \simeq \Sigma_\mathfrak{p}(-2+d_1+d_2) = \Sigma_\mathfrak{p}(\delta).$$

It follows that

$$\left[\mathrm{Ext}^1_{\Sigma_\mathfrak{p}}(\Sigma_\mathfrak{p}/(\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p}),\Sigma_\mathfrak{p})\right]_i \simeq \left[\mathrm{Ext}^1_{\Sigma_\mathfrak{p}}(\Sigma_\mathfrak{p}/(\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p}),\omega_{\Sigma_\mathfrak{p}})\right]_{i-\delta}.$$

As $\dim \Sigma_\mathfrak{p} = 1$, local duality implies that the latter module vanishes if and only if

$$\left[\mathrm{H}^0_\mathfrak{M}(\Sigma_\mathfrak{p}/(\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p}))\right]_{\delta-i} = 0,$$

where $\mathfrak{M}$ denotes the homogeneous maximal ideal of $\Sigma_\mathfrak{p}$. To finish, we note that

$$\left[\mathrm{H}^0_\mathfrak{M}(\Sigma_\mathfrak{p}/(\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p}))\right]_{\delta-i} \subset \left[\Sigma_\mathfrak{p}/(\mathfrak{m}^{\delta-i}\Sigma_\mathfrak{p})\right]_{\delta-i} = 0.$$

□

Corollary 2.14. *Adopt Data* 2.1. *For each integer i, the* $\mathrm{Sym}(I)$*-ideals* $\mathcal{A}_{\geq i}$ *and* $0 :_{\mathrm{Sym}(I)} \mathfrak{m}^{d-1-i}$ *are equal.*

Proof. Assume first that $d-1 \leq i$. In this case, $\mathcal{A}_{\geq i} = 0$ by Corollary 2.5, part (2). On the other hand, in this case, $d-1-i \leq 0$; so, $\mathfrak{m}^{d-1-i} = R$ and $0 :_{\mathrm{Sym}(I)} \mathfrak{m}^{d-1-i} = 0 :_{\mathrm{Sym}(I)} R = 0$.

Now assume that $i \leq 0$. In this case, $\mathcal{A}_{\geq i} = \mathcal{A}$. On the other hand, in this case, $d-1 \leq d-1-i$; thus, $\mathcal{A}\,\mathfrak{m}^{d-1-i} \subset \mathcal{A}_{\geq d-1-i} \subseteq \mathcal{A}_{\geq d-1} = 0$ and

$$\mathcal{A} \subset 0 :_{\mathrm{Sym}(I)} \mathfrak{m}^{d-1-i} \subset 0 :_{\mathrm{Sym}(I)} \mathfrak{m}^\infty = \mathcal{A}.$$

Finally, assume that $1 \leq i \leq d-2$. In this case, $\mathfrak{m}^{d-1-i}\mathcal{A}_{\geq i} \subset A_{\geq d-1} = A_{\geq \delta+1} = 0$, where the last equality holds by part (2) of Corollary 2.5. It follows that $\mathcal{A}_{\geq i} \subset 0 :_{\mathrm{Sym}(I)} \mathfrak{m}^{d-1-i}$. To see the other

inclusion, let $\lambda \neq 0$ be a homogeneous element in $0 :_{\mathrm{Sym}(I)} \mathfrak{m}^{d-1-i}$. Clearly $\lambda \in 0 :_{\mathrm{Sym}(I)} \mathfrak{m}^{\infty} = \mathcal{A}$. We prove that λ has degree at least i. Suppose otherwise. In this case, we observe that

$$\lambda[\mathrm{Sym}(I)]_{d-2-\deg\lambda} \subset \lambda\mathfrak{m}^{d-2-\deg\lambda}\mathrm{Sym}(I) \subset \lambda\mathfrak{m}^{d-1-i}\mathrm{Sym}(I) = 0,$$

where the last inclusion holds because $d-2-\deg\lambda \geq d-1-i$. This shows that λ is a non-zero homogeneous element with the property that multiplication by λ induces the zero homomorphism from $[\mathrm{Sym}(I)]_{d-2-\deg\lambda}$ to $\mathcal{A}_{d-2}$. This contradicts the injectivity of the isomorphism established in Theorem 2.13. $\square$

2.D EXPLICIT S-MODULE GENERATORS FOR $\mathcal{A}_i$, WHEN i IS LARGE.

When i is chosen so that g_1 and g_2 lie in $\mathfrak{m}^{d-1-i}B$, then the equality of Corollary 2.14 has an interpretation in terms of linkage. In the present subsection we exploit that interpretation in order to exhibit explicit generators. Hong, Simis, and Vasconcelos [16, Sect. 3] have used linkage in a similar manner.

Definition 2.15. For each positive index ℓ, define Λ_ℓ to be the $(\ell+1)\times\ell$ matrix

$$\Lambda_\ell = \begin{bmatrix} -x & 0 & 0 & \cdots & 0 & 0 \\ y & -x & 0 & \cdots & 0 & 0 \\ 0 & y & . & \cdots & 0 & 0 \\ 0 & 0 & . & . & 0 & 0 \\ \vdots & \vdots & \vdots & . & . & \vdots \\ 0 & 0 & 0 & \cdots & . & -x \\ 0 & 0 & 0 & \cdots & 0 & y \end{bmatrix}.$$

Notice that Λ_ℓ is a Hilbert-Burch matrix for the row vector $\begin{bmatrix} y^\ell & xy^{\ell-1} & \cdots & x^\ell \end{bmatrix}$.

If ℓ is an index with $1 \leq \ell \leq d_1$, then the polynomials g_1, g_2 of Data 2.1 are both in $(x,y)^\ell B$. Let Ξ_ℓ be an $(\ell+1)\times 2$ matrix of bi-homogeneous elements of B, of bi-degree $(d_1-\ell,1)$ in column 1 and bi-degree $(d_2-\ell,1)$ in column 2, with

(2.15.1) $$\begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} y^\ell & xy^{\ell-1} & \cdots & x^\ell \end{bmatrix} \Xi_\ell$$

and Ψ_ℓ be the $(\ell+1)\times(\ell+2)$ matrix

(2.15.2) $$\Psi_\ell = \begin{bmatrix} \Lambda_\ell & \Xi_\ell \end{bmatrix}$$

of bi-homogeneous elements of B.

Corollary 2.16. *Adopt Data* 2.1. *Let i be an index with $d_2 - 1 \leq i \leq \delta$ and Ψ_{d-1-i} be a matrix as described in* (2.15.2)*.*

(1) *The ideals*

$$\mathcal{A}_{\geq i}, \quad \mathcal{A}_{(i,2)}\mathrm{Sym}(I), \quad \textit{and} \quad \mathrm{I}_{d-i}(\Psi_{d-1-i})\mathrm{Sym}(I)$$

of $\mathrm{Sym}(I)$ *are equal. In particular, the minimal bi-homogeneous generators of the* $\mathrm{Sym}(I)$*-ideal $\mathcal{A}$ all have $\{x,y\}$-degree at most d_2-1.*

(2) *The S-module $\mathcal{A}_i$ is free of rank $d-i-1$; a basis consists of the maximal minors of the matrix Ψ_{d-1-i} that involve the last two columns.*

Note. The bound on the generator degrees that is given in item (1) is also given in [3, Thm. 4.6].

Proof. Recall that $\mathcal{A}_{\geq i} = 0 :_{\mathrm{Sym}(I)} \mathfrak{m}^{d-1-i}$ by Corollary 2.14. The latter ideal equals

$$\frac{(g_1,g_2)B :_B \mathfrak{m}^{d-1-i}}{(g_1,g_2)B}.$$

The homogeneous B-ideal $\mathfrak{m}^{d-1-i}B$ is perfect of grade 2 and it contains the homogeneous B-regular sequence g_1,g_2 because the hypothesis $d_2-1 \leq i \leq \delta$ guarantees that

$$1 \leq d-1-i \leq d_1. \tag{2.16.1}$$

The matrix Λ_{d-1-i} of Definition 2.15 is a homogeneous matrix of relations among the generators $y^{d-1-i}, xy^{d-2-i}, \cdots, x^{d-1-i}$ of $\mathfrak{m}^{d-1-i}B$, and Ξ_{d-1-i} is a homogeneous matrix of coefficients when writing g_1,g_2 in terms of these generators. The inequalities of (2.16.1) guarantee that the matrix Ξ_{d-1-i} of (2.15.1) is defined. Now, according to [9], the linked ideal $(g_1,g_2)B :_B \mathfrak{m}^{d-1-i}$ is generated by the maximal minors of the $d-i$ by $d-i+1$ matrix $\Psi_{d-1-i} = \begin{bmatrix}\Lambda_{d-1-i} & \Xi_{d-1-i}\end{bmatrix}$. Thus indeed,

$$\frac{(g_1,g_2)B :_B \mathfrak{m}^{d-1-i}}{(g_1,g_2)B} = \frac{\mathrm{I}_{d-i}(\Psi_{d-1-i})B}{(g_1,g_2)B} = \mathrm{I}_{d-i}(\Psi_{d-1-i})\mathrm{Sym}(I).$$

The maximal minors of Ψ_{d-1-i} are g_1,g_2 (up to sign), together with the minors $\Delta_1,\dots,\Delta_{d-1-i}$ that involve the last two columns. We conclude that

$$\mathcal{A}_{\geq i} = (\Delta_1,\dots,\Delta_{d-1-i})\mathrm{Sym}(I)$$

Observe that each Δ_i has bi-degree $(i,2)$. This completes the proof of (1).

Assertion (2) follows from Theorem 2.7, part (3), and the fact that the elements $\Delta_1,\dots,\Delta_{d-1-i}$ generate $\mathcal{A}_i$ as an S-module. □

Remark 2.17. Corollary 2.16 describes $\mathcal{A}_{i_0}$ for each i_0 with $d_2-1 \leq i_0 \leq \delta$. We highlight the content of Corollary 2.16 at the boundaries $i_0 = d_2-1$ and $i_0 = \delta$.

Take $i_0 = \delta$. In this situation, the S-module $\mathcal{A}_\delta$ is free of rank 1 with basis element any Sylvester form "syl", where syl is the determinant of any fixed 2×2 matrix Ξ_1 described in (2.15.1). (The k-vector space $\mathcal{A}_{(\delta,2)}$ is one-dimensional and we use the name "syl" for any basis element of this vector space.) Notice that the entries of column m of Ξ_1 are homogeneous of bi-degree $(d_m-1,1)$ and syl is homogeneous of bi-degree $(\delta,2)$. Let i be arbitrary. One explicit realization of the isomorphism

$$\mathcal{A}_i \simeq \mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i}, S(-2))$$

of Corollary 2.5 is that this isomorphism is induced by the composition

$$\mathcal{A}_i \otimes \mathrm{Sym}(I)_{\delta-i} \xrightarrow{\text{mult}} \mathcal{A}_\delta \xrightarrow{\sigma} S(-2), \tag{2.17.1}$$

where mult is the multiplication in $\mathrm{Sym}(I)$, as described in Theorem 2.13, and σ is the inverse of the isomorphism $S(-2) \to \mathcal{A}_\delta$ which sends 1 to syl.

Take $i_0 = d_2 - 1$. Corollary 2.16 guarantees that $\mathcal{A}_{\geq d_2-1}$ is generated as a B-module by the maximal minors of $\Psi_{d_1} = [\Lambda_{d_1}, \Xi_{d_1}]$ which involve both columns of Ξ_{d_1}. Each of these minors is homogeneous of bi-degree $(d_2-1, 2)$, and one choice for Ξ_{d_1} is

$$\begin{bmatrix} c_{0,1} & c_{0,2}y^{d_2-d_1} + \cdots + c_{d_2-d_1,2}x^{d_2-d_1} \\ c_{1,1} & c_{d_2-d_1+1,2}x^{d_2-d_1} \\ \vdots & \vdots \\ c_{d_1,1} & c_{d_2,2}x^{d_2-d_1} \end{bmatrix},$$

for $c_{\ell,m}$ as described in (2.6.1).

3. The case of a generalized zero in the first column of φ.

Data 3.1. Adopt Data 2.1 with $d_1 < d_2$. Assume that there is a generalized zero in the first column of φ.

In this section we are in the situation of Data 3.1 and we describe $\mathcal{A}_{\geq d_1-1}$ as an S-module and as a B-module. The hypothesis that there is a generalized zero in the first column of φ has geometric significance. Indeed, as described in Remark 2.9, the homogeneous minimal generators of I describe a morphism $\eta : \mathbb{P}^1_k \to \mathbb{P}^2_k$ whose image is a rational curve $\mathcal{C}$. If the morphism η is birational onto $\mathcal{C}$ (or equivalently, if the curve $\mathcal{C}$ has degree d), then the assumption that φ has a generalized zero in the first column is equivalent to the assumption that $\mathcal{C}$ has a singularity of multiplicity d_2; see the General Lemma in [6, Cor. 1.9]; or [26, Thm. 3] or [5, Thm. 1].

The S-module structure of $\mathcal{A}_{\geq d_1-1}$ is completely described in Theorem 3.3: $\mathcal{A}_{\geq d_1-1}$ is a free S-module of finite rank and a complete list of the bi-degrees of an S-module basis for $\mathcal{A}_{\geq d_1-1}$ is given; see also Table 3.5.1. The B-module structure of $\mathcal{A}_{\geq d_1-1}$ is described in Corollary 3.10

The part of $\mathrm{Sym}(I)$ that corresponds to $\mathcal{A}_{\geq d_1-1}$, under the duality of Theorem 2.4, is $\mathrm{Sym}(I)_{\leq d_2-1}$. There is no contribution from g_2 to the S-module $\mathrm{Sym}(I)_{\leq d_2-1}$ in the bi-homogeneous B-resolution of $\mathrm{Sym}(I)$. So, basically, we may ignore g_2 in the present section. Furthermore, the hypothesis that the first column of φ has a generalized zero allows us to make the critical calculation over a subring U of S, where U is a polynomial ring in two variables. In the proof of Theorem 3.3, we decompose various bi-graded complexes over $R \otimes_k U$ into their R-graded components and their U-graded components. Ultimately, the critical calculation is to produce a lower bound for the degrees of the syzygies of a U-module homomorphism.

Lemma 3.2 is a statement about U-module homomorphisms. This lemma explains how, sometimes, lower bounds for the degrees of syzygies suffice to determine these degrees.The statement of Lemma 3.2 may be deduced from a classification of matrices whose entries are linear forms from U. This classification was known by and Weierstrass (in the singular case) and Kronecker (in the general case); see [10, Chapt XII]. The proof we give for Lemma 3.2 uses Hilbert series to relate the twists in a homogeneous resolution to the betti numbers. This technique, which is now standard, was

introduced by Peskine and Szpiro [25] and was used with great success by Herzog and Kühl [12]; it was also a motivation for Boij-Söderberg theory.

Lemma 3.2. *Let M be a graded module of finite length with a linear presentation over the polynomial ring $U = k[T_1,T_2]$. If a homogeneous resolution of M has the form*

$$0 \to \bigoplus_{\ell=1}^{m} U(-b_\ell) \to U(-1)^n \to U^{n-m} \to M \to 0,$$

then $\sum_{\ell=1}^m b_\ell = n$.

Proof. The Hilbert series of M is $h_M(t)/(1-t)^2$, where $h_M(t) = n-m-nt+\sum_{\ell=1}^m t^{b_\ell}$. Since M has finite length, this series is a polynomial; hence, $(1-t)^2$ divides $h_M(t)$. Therefore, $h'_M(1) = 0$, which gives the assertion. $\square$

Theorem 3.3. *Adopt Data* 3.1. *If $d_1 - 1 \le i \le d_2 - 1$, then*

$$\mathcal{A}_i \simeq \bigoplus_{\ell=1}^{d_1} S(-a_\ell),$$

where

$$\left\lfloor \frac{d+d_1-1-i}{d_1} \right\rfloor = a_1 \le \cdots \le a_{d_1} = \left\lceil \frac{d+d_1-1-i}{d_1} \right\rceil$$

and

$$\sum_{\ell=1}^{d_1} a_\ell = d+d_1-1-i.$$

Remark 3.4. We offer an alternate phrasing for Theorem 3.3. If $d_1 - 1 \le i \le d_2 - 1$ and α_i and β_i are integers with

$$d+d_1-1-i = \alpha_i d_1 + \beta_i \quad \text{and} \quad 0 \le \beta_i \le d_1 - 1,$$

then

$$\mathcal{A}_i \simeq S(-\alpha_i)^{d_1-\beta_i} \oplus S(-\alpha_i - 1)^{\beta_i}.$$

Of course, in this language, α_i is equal to $\left\lfloor \frac{d+d_1-1-i}{d_1} \right\rfloor$ and β_i is the "remainder that is obtained when $d+d_1-1-i$ is divided by d_1". Observe that the parameter α_i is always at least 2, the exponent $d_1 - \beta_i$ is positive, and the other exponent, β_i, is non-negative.

Proof of Theorem 3.3. The case $i = d_2 - 1$ is covered in part (3) of Corollary 2.7. Fix an integer i with

$$d_1 - 1 \le i \le d_2 - 2.$$

We prove the following four ingredients.

(3.4.1) The S-module $\mathcal{A}_i$ is free of rank d_1.

Once (3.4.1) is established, then we define the shifts a_ℓ by $\mathcal{A}_i \simeq \bigoplus_{\ell=1}^{d_1} S(-a_\ell)$ and we prove

$$\sum_{\ell=1}^{d_1} a_\ell = d+d_1-1-i, \tag{3.4.2}$$

$$\mathcal{A}_{(i,j)} = 0 \quad \text{for } j \le \alpha_i - 1, \quad \text{and} \tag{3.4.3}$$

$$\dim_k \mathcal{A}_{(i,\alpha_i)} = d_1 - \beta_i, \tag{3.4.4}$$

where we have used the language of Remark 3.4. Once the four ingredients have been established, then it is not difficult to complete the proof. If $\beta_i = 0$, then the result follows immediately from (3.4.1) and (3.4.4); and if $0 < \beta_i$, then one may apply the pigeon hole principle. Indeed, if (3.4.1) – (3.4.4) hold and $0 < \beta_i$, then

$$\alpha_i + 1 \le a_{d_1-\beta_i+1} \le \cdots \le a_{d_1}$$

and

$$\begin{aligned}(\alpha_i+1)\beta_i \le a_{d_1-\beta_i+1} + \cdots + a_{d_1} &= \sum_{\ell=1}^{d_1} a_\ell - (d_1-\beta_i)\alpha_i = (d+d_1-1-i) - (d_1-\beta_i)\alpha_i \\ &= (\alpha_i d_1 + \beta_i) - (d_1-\beta_i)\alpha_i = (\alpha_i+1)\beta_i;\end{aligned}$$

hence, $a_\ell = \alpha_i + 1$ for $d_1 - \beta_i + 1 \le \ell \le d_1$.

The assertion about the rank of $\mathcal{A}_i$ is established in part (2) of Theorem 2.7.

The hypothesis that the first column of φ has a generalized zero ensures that after performing row operations on φ and renaming the generators of S_1, we have that φ has the form

$$\varphi = \begin{pmatrix} f_1 & * \\ f_2 & * \\ 0 & * \end{pmatrix}$$

and g_1, which is equal to $[T_1, T_2, T_3]$ times the first column of φ, only involves T_1 and T_2. Indeed, $g_1 = f_1T_1 + f_2T_2$. The hypothesis from Data 2.1 that I has height two guarantees that f_1 and f_2 are a regular sequence of forms of degree d_1 in R. At this point, we introduce the subrings $U = k[T_1, T_2]$ of S and $C = R \otimes_k U$ of $R \otimes_k S = B$. Notice that $C[T_3] = B$ and g_1 is in C.

Let $K_\bullet$ be the following bi-graded complex of $C[T_3]$-modules:

$$K_\bullet : \quad C[T_3](-d_1,-1) \xrightarrow{g_1} C[T_3] \longrightarrow \mathrm{Sym}(I) \to 0. \tag{3.4.5}$$

The graded strand $[K_\bullet]_{(\delta-i,\_)}$ of $K_\bullet$ is exact because $\delta - i \le d_2 - 1$ by our assumption $d_1 - 1 \le i$. Taking graded S-duals we obtain the complex

$$\underline{\mathrm{Hom}}_S(K_\bullet, S): \quad 0 \to \underline{\mathrm{Hom}}_S(\mathrm{Sym}(I), S) \longrightarrow \underline{\mathrm{Hom}}_S(C[T_3], S) \xrightarrow{g_1} \underline{\mathrm{Hom}}_S(C[T_3], S)(d_1, 1).$$

Furthermore, the graded strand $\underline{\mathrm{Hom}}_S(K_\bullet, S)_{(i-\delta,\_)}$ is exact. Theorem 2.4 guarantees that

$$\underline{\mathrm{Hom}}_S(\mathrm{Sym}(I), S) \simeq \mathcal{A}(\delta, 2). \tag{3.4.6}$$

Observe that

$$\underline{\mathrm{Hom}}_S(C[T_3], S) \simeq \underline{\mathrm{Hom}}_S(R \otimes_k S, S) = \underline{\mathrm{Hom}}_k(R, k) \otimes_k S \simeq \underline{\mathrm{Hom}}_k(R, k) \otimes_k U \otimes_k k[T_3]. \tag{3.4.7}$$

Combine (3.4.6) and (3.4.7) to see that the complex $\underline{\mathrm{Hom}}_S(K_\bullet, S)$ may be identified with

$$0 \to \mathcal{A}(\delta, 2) \to \underline{\mathrm{Hom}}_k(R, k) \otimes_k U \otimes_k k[T_3] \xrightarrow{g_1} \underline{\mathrm{Hom}}_k(R, k)(d_1) \otimes_k U(1) \otimes_k k[T_3].$$

Since $g_1 \in R \otimes_k U$, multiplication by g_1 gives a bi-homogeneous $(R \otimes_k U)$- module homomorphism

$$\psi : \underline{\mathrm{Hom}}_k(R, k) \otimes_k U \xrightarrow{g_1} \underline{\mathrm{Hom}}_k(R, k)(d_1) \otimes_k U(1).$$

We focus on the kernel and cokernel of various R-graded and U-graded components of ψ. We have

$$(\ker\psi)_{i-\delta}(-2)[T_3] \simeq \mathcal{A}_i, \quad \text{for } d_1-1 \le i \le d_2-2. \tag{3.4.8}$$

It follows immediately that $\mathcal{A}_i$ is free over S because $\ker\psi_{i-\delta}$ is a second syzygy over U, a polynomial ring in 2 variables. Thus, (3.4.1) is established. Notice that $(\ker\psi)_{i-\delta} \simeq \bigoplus_{\ell=1}^{d_1} U(-a_\ell+2)$. The left-most map in the complex $K_\bullet$ of (3.4.5) is injective; consequently, a calculation similar to the one that produced (3.4.8) yields

$$(\operatorname{coker}\psi)_{i-\delta}[T_3] \simeq \operatorname{Ext}^1_S(\operatorname{Sym}(I)_{\delta-i}, S) \quad \text{for } d_1-1 \le i \le d_2-2.$$

According to part (1.a) of Theorem 2.11, $\operatorname{Ext}^1_S(\operatorname{Sym}(I)_{\delta-i}, S)$ vanishes locally in codimension one whenever $d_1-1 \le i \le d_2-2$; hence, $(\operatorname{coker}\psi)_{i-\delta}$ has finite length as a U-module. Now we can apply Lemma 3.2 to $(\operatorname{coker}\psi)_{i-\delta}(-1)$, which has a homogeneous free U-resolution of the form:

$$0 \to (\ker\psi)_{i-\delta}(-1) \to \underline{\operatorname{Hom}}_k(R,k)_{i-\delta} \otimes_k U(-1) \to \underline{\operatorname{Hom}}_k(R,k)_{i-\delta+d_1} \otimes_k U \to (\operatorname{coker}\psi)_{i-\delta}(-1) \to 0.$$

As $(\ker\psi)_{i-\delta}(-1) \simeq \bigoplus_{\ell=1}^{d_1} U(-a_\ell+1)$ and

$$\dim_k \underline{\operatorname{Hom}}_k(R,k)_{i-\delta} = \dim_k R_{\delta-i} = \delta+1-i,$$

Lemma 3.2 gives that $\sum_{\ell=1}^{d_1}(a_\ell-1) = \delta+1-i$; or equivalently, $\sum_{\ell=1}^{d_1} a_\ell = d+d_1-1-i$. The second ingredient, (3.4.2), has been established.

We re-phrase (3.4.8) as

$$\mathcal{A}_{(i,j)} \simeq (\ker\psi_{(i-\delta,j-2)})[T_3], \quad \text{for } d_1-1 \le i \le d_2-2 \text{ and all } j, \tag{3.4.9}$$

in order to focus on the individual components $\mathcal{A}_{(i,j)}$ of the free S-module $\mathcal{A}_i$. Each such component is a finite dimensional k-vector space. We see that

$$\begin{array}{llllll} \text{Claim (3.4.3) holds} & \iff & \mathcal{A}_{(i,j)} = 0 & \text{for} & j & \le \left\lfloor \frac{d-1-i}{d_1} \right\rfloor \\ & \iff & (\ker\psi)_{(i-\delta,j-2)} = 0 & \text{for} & j-2 & \le \left\lfloor \frac{d-1-i}{d_1} \right\rfloor - 2 \\ & \iff & (\ker\psi)_{(m,n)} = 0 & \text{for} & n & \le \left\lfloor \frac{1-m}{d_1} \right\rfloor - 2. \end{array}$$

To prove the most recent version of (3.4.3) we fix an integer n and consider the homogeneous R-module homomorphism

$$\psi_{(\_\_,n)} : \underline{\operatorname{Hom}}_k(R,k) \otimes_k U_n \longrightarrow \underline{\operatorname{Hom}}_k(R,k)(d_1) \otimes_k U_{n+1}.$$

Recall that ψ is multiplication by $f_1T_1 + f_2T_2$. We choose the k-bases $T_1^n, \dots, T_2^n$ and $T_1^{n+1}, \dots, T_2^{n+1}$ for U_n and U_{n+1}, respectively and we see that the R-module homomorphism $\psi_{(\_\_,n)}$ is represented by

the $(n+2)\times(n+1)$ matrix

$$\begin{bmatrix} f_1 & 0 & 0 & \cdots & 0 \\ f_2 & f_1 & 0 & \cdots & 0 \\ 0 & f_2 & f_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & f_2 & f_1 \\ 0 & \cdots & 0 & 0 & f_2 \end{bmatrix}.$$

Take the graded k-dual of $\psi_{(\_,n)}$ to obtain a homogeneous R-module homomorphism

$${}_n\chi : R(-d_1)\otimes_k U^*_{n+1} \longrightarrow R\otimes_k U^*_n,$$

given by the $(n+1)\times(n+2)$ matrix

$$\begin{bmatrix} f_1 & f_2 & 0 & \cdots & 0 \\ 0 & f_1 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & f_1 & f_2 \end{bmatrix},$$

with entries in R, where $(\_)^*$ means $\mathrm{Hom}_k(\_,k)$. (The name ${}_n\chi$ emphasizes that this map depends on n; however, this map is not created as a graded strand of some other, previously named, object.) Notice that $\psi_{(m,n)}$ is injective if and only if the graded component ${}_n\chi_{-m}$ of ${}_n\chi$ is surjective; moreover, in general,

$$\dim_k \ker \psi_{(m,n)} = \dim_k \operatorname{coker} {}_n\chi_{-m}. \tag{3.4.10}$$

The Buchsbaum-Rim complex yields a homogeneous free resolution of $\operatorname{coker} {}_n\chi$:

$$0 \to R(-(n+2)d_1) \to R(-d_1)\otimes_k U^*_{n+1} \xrightarrow{{}_n\chi} R\otimes_k U^*_n. \tag{3.4.11}$$

This resolution shows that the socle of $\operatorname{coker} {}_n\chi$ is concentrated in degree $(n+2)d_1-2$. Thus, ${}_n\chi_{-m}$ is surjective if $-m \geq (n+2)d_1-1$. However $-m\geq (n+2)d_1-1 \iff n\leq \frac{1-m}{d_1}-2$, and therefore Claim 3.4.3 holds.

We now compute $\dim_k \mathcal{A}_{(i,\alpha_i)}$, as required by Claim (3.4.4). The generators of the free S-module $\mathcal{A}_i$ all have degree at least α_i. Therefore, the number of minimal generators of degree α_i for the S-module $\mathcal{A}_i$ is the dimension of the k-vector space $\mathcal{A}_{(i,\alpha_i)}$, and, by (3.4.9), the vector spaces $\mathcal{A}_{(i,\alpha_i)}$ and $\ker \psi_{(i-\delta,\alpha_i-2)}$ have the same dimension. In this calculation, we dig more deeply into the sequence of ideas that was used in the proof of (3.4.3). In particular, (3.4.9) and (3.4.10) give

$$\dim_k \mathcal{A}_{(i,\alpha_i)} = \dim_k \ker \psi_{(i-\delta,\alpha_i-2)} = \dim_k(\operatorname{coker}({}_{\alpha_i-2}\chi_{\delta-i})). \tag{3.4.12}$$

We again use (3.4.11) to make this computation:

$$\begin{aligned}\dim_k(\operatorname{coker}({}_{\alpha_i-2}\chi_{\delta-i})) = \dim_k(R\otimes_k U^*_{\alpha_i-2})_{\delta-i} - \dim_k(R(-d_1)\otimes_k U^*_{\alpha_i-1})_{\delta-i} \\ + \dim_k R(-\alpha_i d_1)_{\delta-i}.\end{aligned} \tag{3.4.13}$$

We notice that

$$(R(-d_1)\otimes_k U^*_{\alpha_i-1})_{\delta-i} \neq 0 \quad \text{and} \tag{3.4.14}$$

$$R(-\alpha_i d_1)_{\delta-i} = 0, \tag{3.4.15}$$

for $d_1-1\le i\le d_2-2$. Indeed, the hypothesis $i\le d_2-2$ guarantees that $0\le d_2-2-i=\delta-d_1-i$; hence, $d_1\le\delta-i$ and (3.4.14) holds. Also, $a-(b-1)\le\lfloor\frac{a}{b}\rfloor b$ for all positive integers a and b; hence,

$$\delta-i<d-i=(d+d_1-1-i)-(d_1-1)\le\left\lfloor\frac{d+d_1-1-i}{d_1}\right\rfloor d_1=\alpha_i d_1$$

and (3.4.15) holds. Combine (3.4.12), (3.4.13), (3.4.14), and (3.4.15) to see that

$$\begin{array}{rcl}\dim_k\mathcal{A}_{(i,\alpha_i)}=\dim_k(\operatorname{coker}({}_{\alpha_i-2}\chi_{\delta-i})) &=& \dim_k(R\otimes_k U^*_{\alpha_i-2})_{\delta-i}-\dim_k(R(-d_1)\otimes_k U^*_{\alpha_i-1})_{\delta-i}\\ &=& (\delta-i+1)(\alpha_i-1)-(\delta-i+1-d_1)\alpha_i\\ &=& -(\delta-i+1)+d_1\alpha_i\\ &=& -(d-1-i)+d_1\alpha_i\\ &=& d_1-(d+d_1-1-i)+d_1\alpha_i\\ &=& d_1-(d_1\alpha_i+\beta_i)+d_1\alpha_i\\ &=& d_1-\beta_i,\end{array}$$

which completes the proof of (3.4.4). All four claims (3.4.1) – (3.4.4) have been established. The proof is complete. □

Table 3.5. Adopt Data 3.1. Table 3.5.1 records the S-module structure of

$$\mathcal{A}_{\ge d_1-1}\simeq\bigoplus S(-(i,j))^{n_{i,j}}.$$

The exponent $n_{i,j}$ sits in position (i,j), where i is plotted on the horizontal axis and j is plotted on the vertical axis. In other words, a minimal homogeneous basis for the free S-module $\mathcal{A}_i$ has $n_{i,j}$ generators of bi-degree (i,j).

We describe, in words, the transition of the generator degrees **to** $\mathcal{A}_{i-1}$ **from** $\mathcal{A}_i$, beginning at the right side of the table. Recall that $\mathcal{A}_i=0$ for $\delta+1\le i$.

If $d_2-1\le i\le\delta$, then the generators of $\mathcal{A}_i$ are concentrated in the unique degree 2. The rank of $\mathcal{A}_\delta$ is 1 and if $d_2-1\le i\le\delta-1$, then $\operatorname{rank}\mathcal{A}_i=\operatorname{rank}\mathcal{A}_{i+1}+1$. The relevant proof for this part of the table is contained in Corollary 2.16. Theorem 3.3 contains the proof for the range $d_1-1\le i\le d_2-1$. In this range the rank of $\mathcal{A}_i$ remains constant at d_1 and the generators of $\mathcal{A}_i$ live in two degrees, or, occasionally, only one degree. As one looks from right to left, one free rank one summand of **lowest** shift in $\mathcal{A}_i$ is replaced by a free rank one summand with shift **one** higher in $\mathcal{A}_{i-1}$.

One continues this pattern all the way until the left boundary of $\mathcal{A}_{\ge d_1-1}$; namely $i=d_1-1$. The shifts in $\mathcal{A}_{d_1-1}$ are given by $\lfloor\frac{d}{d_1}\rfloor:(d_1-r)$ and $(\lfloor\frac{d}{d_1}\rfloor+1):r$ where r is the remainder of d upon division by d_1; that is, $d=d_1\lfloor\frac{d}{d_1}\rfloor+r$, with $0\le r\le d_1-1$, and

$$\mathcal{A}_{d_1-1}=S(-\lfloor\tfrac{d}{d_1}\rfloor)^{d_1-r}\oplus S(-\lfloor\tfrac{d}{d_1}\rfloor-1)^r.$$

Remark 3.6. The “exterior corner points” (i,j) in Table 3.5.1 are very important when one considers the B-module structure of $\mathcal{A}_{\ge d_1-1}$ because degree considerations show that the corresponding basis element of the k-vector space $\mathcal{A}_{(i,j)}$ is part of a minimal bi-homogeneous generating set for the B-module $\mathcal{A}_{\ge d_1-1}$. For this reason we carefully record where these corner points occur. Continue to write $d=d_1\lfloor\frac{d}{d_1}\rfloor+r$, with $0\le r\le d_1-1$. Let (i,j) be integers. We claim that

$$d_1\le i\le d_2-1\quad\text{and}\quad\mathcal{A}_i=S(-j)^1\oplus S(-j-1)^{d_1-1}\tag{3.6.1}$$

T-deg													
$\lfloor \frac{d}{d_1} \rfloor + 1$	r	$\cdots$											
$\lfloor \frac{d}{d_1} \rfloor$	d_1-r	$\cdots$											
$\vdots$													
$\lfloor \frac{d}{d_1} \rfloor - \lambda + 1$		$\cdots$	d_1	d_1-1	d_1-2	$\cdots$							
$\lfloor \frac{d}{d_1} \rfloor - \lambda$				1^*	2	$\cdots$							
$\vdots$													
3						$\cdots$	1						
2						$\cdots$	d_1-1	d_1	d_1-1	d_1-2	$\cdots$	1	
	d_1-1	$\cdots$	λd_1+r-1	λd_1+r	λd_1+r+1	$\cdots$	d_2-2	d_2-1	d_2	d_2+1	$\cdots$	δ	xy-deg

Table 3.5.1. The generator degrees for the free S-module $\mathcal{A}_{\geq d_1-1}$. *This table is described in words in Table 3.5. (Also, we call the 1^* that appears in position $(i,j) = (\lambda d_1 + r, \lfloor \frac{d}{d_1} \rfloor - \lambda)$ an "exterior corner point". We never refer to any entry in the left-most column as an exterior corner point. In Remark* 3.6 *we calculate that the exterior corner points occur when* $1 \leq \lambda \leq \lfloor \frac{d}{d_1} \rfloor - 2$.)

if and only if $(i,j) = (\lambda d_1 + r, \lfloor \frac{d}{d_1} \rfloor - \lambda)$ for some integer λ with $1 \leq \lambda \leq \lfloor \frac{d}{d_1} \rfloor - 2$. (Notice that we never refer to any entry in the left-most column of Table 3.5.1 as an exterior corner point.)

Proof. Fix an integer j. Apply Theorem 3.3, by way of Remark 3.4, to see that

$$\begin{aligned} \text{(3.6.1) holds} \quad &\iff \quad d_1 \leq i \leq d_2 - 1 \quad \text{and} \quad d + d_1 - 1 - i = jd_1 + d_1 - 1 \\ &\iff \quad d_1 \leq i \leq d_2 - 1 \quad \text{and} \quad d - i = jd_1. \end{aligned}$$

Write $d = d_1 \lfloor \frac{d}{d_1} \rfloor + r$ to see that

$$\begin{aligned} \text{(3.6.1) holds} \quad &\iff \quad d_1 \leq i \leq d_2 - 1 \quad \text{and} \quad d_1 \lfloor \tfrac{d}{d_1} \rfloor + r - i = d_1 j \\ &\iff \quad d_1 \leq i \leq d_2 - 1 \quad \text{and} \quad \lfloor \tfrac{d}{d_1} \rfloor - \tfrac{i-r}{d_1} = j. \end{aligned}$$

Let $\lambda = \frac{i-r}{d_1}$. It follows that

$$\text{(3.6.1) holds} \quad \iff \quad \begin{cases} \text{there exists an integer } \lambda \text{ with } i = \lambda d_1 + r, \; j = \lfloor \frac{d}{d_1} \rfloor - \lambda, \text{ and} \\ d_1 \leq \lambda d_1 + r \leq d_2 - 1. \end{cases}$$

To complete the proof we show that

$$d_1 \leq \lambda d_1 + r \leq d_2 - 1 \iff 1 \leq \lambda \leq \lfloor \tfrac{d}{d_1} \rfloor - 2.$$

The left hand inequalities $d_1 \leq \lambda d_1 + r$ and $1 \leq \lambda$ are equivalent because $0 \leq r \leq d_1 - 1$. To see that

$$\lambda d_1 + r \leq d_2 - 1 \iff \lambda \leq \lfloor \tfrac{d}{d_1} \rfloor - 2,$$

add $d_1 - r$ to both sides of the left hand inequality and use $d_1 + d_2 - r = d_1 \lfloor \frac{d}{d_1} \rfloor$ to see that

$$\lambda d_1 + r \leq d_2 - 1 \iff \lambda d_1 + d_1 \leq d_1 \lfloor \tfrac{d}{d_1} \rfloor - 1 \iff \lambda \leq \lfloor \tfrac{d}{d_1} \rfloor - \tfrac{1}{d_1} - 1 \iff \lambda \leq \lfloor \tfrac{d}{d_1} \rfloor - 2.$$

□

Remark 3.7. In the notation of Table 3.5, we see that

(1) $\mathcal{A}_{(d_1-1,\lfloor\frac{d}{d_1}\rfloor)}$ is a k-vector space of dimension d_1-r,
(2) $\mathcal{A}_{(d_1-1,\lfloor\frac{d}{d_1}\rfloor+1)}$ is a k-vector space of dimension r, and
(3) if λ is an index with $1\le\lambda\le\lfloor\frac{d}{d_1}\rfloor-2$, then $\mathcal{A}_{(\lambda d_1+r,\lfloor\frac{d}{d_1}\rfloor-\lambda)}$ is a k-vector space of dimension 1.

Definition 3.8. Let $v_1,\dots,v_r$ be a k-basis for $\mathcal{A}_{(d_1-1,\lfloor\frac{d}{d_1}\rfloor+1)}$; $u_1,\dots,u_{d_1-r}$ be a k-basis for $\mathcal{A}_{(d_1-1,\lfloor\frac{d}{d_1}\rfloor)}$; for each λ, with $1\le\lambda\le\lfloor\frac{d}{d_1}\rfloor-2$, let w_λ be a k-basis for $\mathcal{A}_{(\lambda d_1+r,\lfloor\frac{d}{d_1}\rfloor-\lambda)}$.

In Corollary 3.10 we prove that the elements of Definition 3.8 form a minimal bi-homogeneous generating set for the B-module $\mathcal{A}_{\ge d_1-1}$. In the proof of Corollary 3.10 we appeal to the following fact about Hilbert functions that may well be known to experts.

Lemma 3.9. *Let k be an algebraically closed field, R a positively graded k-algebra with* $\dim_k R_1>1$, *M a graded R-module and $i\ge 0$ a fixed integer. Assume that for every non-zero element $\ell\in R_1$ the map $M_i\to M_{i+1}$ induced by multiplication with ℓ is injective. Then for any non-zero finite dimensional k-subspace V of M_i one has* $\dim_k R_1V>\dim_k V$.

Proof. Suppose that $\dim_k R_1V\le\dim_k V$. Choose two k-linearly independent elements x and y in R_1. Since $\dim_k xV=\dim_k yV=\dim_k V\ge\dim_k R_1V$, it follows that $xV=R_1V=yV$. Let $e_1,\dots,e_n$ be a k-basis of V. One has

$$x[e_1,\dots,e_n]=y[e_1,\dots,e_n]\cdot\Phi \tag{3.9.1}$$

for some $n\times n$ matrix Φ with entries in k. Let $v\in k^n$ be a non-zero eigenvector of Φ belonging to an eigenvalue $\lambda\in k$ and write $z=[e_1,\dots,e_n]\cdot v$. Notice that z is a non-zero element of M_i. Multiplying equation (3.9.1) by v from the right we obtain $xz=\lambda yz$. Thus $(x-\lambda y)z=0$ in M_{i+1}. Since $x-\lambda y$ is a non-zero element in R_1, we have a contradiction to our assumption on M. □

Corollary 3.10. *Adopt Data 3.1. The set $\{v_1,\dots,v_r\}\cup\{u_1,\dots,u_{d_1-r}\}\cup\{w_\lambda\mid 1\le\lambda\le\lfloor\frac{d}{d_1}\rfloor-2\}$ of elements from Definition* 3.8 *is a minimal bi-homogeneous generating set for the B-module $\mathcal{A}_{\ge d_1-1}$.*

Proof. We may harmlessly assume that k is algebraically closed. Write $X=\{v_\varepsilon\}\cup\{u_\mu\}\cup\{w_\lambda\}$ for the set described in the statement. First notice that no element of X is a B-linear combination of the others. This is a consequence of following facts: $\mathcal{A}_{\ge d_1-1}$ is concentrated in R-degrees $\ge d_1-1$, $\{v_\varepsilon\}\cup\{u_\mu\}$ forms a homogeneous S-basis of $\mathcal{A}_{d_1-1}$ concentrated in S-degrees $\ge\lfloor\frac{d}{d_1}\rfloor$ (see Theorem 3.3), the w_λ have R-degrees $>d_1-1$ and S-degrees $<\lfloor\frac{d}{d_1}\rfloor$, and as λ increases the R-degrees of the w_λ increase strictly whereas their S-degrees decrease strictly.

Thus it suffices to show that $\mathcal{A}_{\ge d_1-1}=XB$. To do so we prove by induction on $i\ge d_1-1$ that

$$\mathcal{A}_i\subset\mathcal{A}_{i-1}R_1+XB. \tag{3.10.1}$$

Notice that from part (1) of Corollary 2.16 we have $\mathcal{A}_{\ge d_2}\subset\mathcal{A}_{d_2-1}R$. Hence we may assume that i is in the range $d_1-1\le i\le d_2-1$. For $i=d_1-1$ equation (3.10.1) follows from Theorem 3.3. As for the induction step let $i\ge d_1$. From Table 3.5.1 we know that $\mathcal{A}_{i-1}$ is a graded free S-module with t

homogeneous basis elements in degree j and $d_1 - t$ homogeneous basis elements in the next higher degree $j+1$, where t is in the range $0 \le t \le d_1 - 1$. Write U for the k-subspace of $\mathcal{A}_{(i-1,j)}$ spanned by the basis elements of degree j and V for the k-subspace of $\mathcal{A}_{(i-1,j+1)}$ spanned by the basis elements of degree $j+1$. Likewise, the graded S-module $\mathcal{A}_i$ has $t+1$ homogeneous basis elements of degree j and $d_1 - t - 1$ homogeneous basis elements of degree $j+1$. These basis elements span k-subspaces $W \subset \mathcal{A}_{(i,j)}$ and $Z \subset \mathcal{A}_{(i,j+1)}$, respectively. Notice that $W = \mathcal{A}_{(i,j)}$. Finally, write Y for the k-subspace of $\mathcal{A}_{(i,j)}$ spanned by the elements w_λ of bi-degree (i,j). Accordingly it suffices to prove that

$$R_1U + Y = W \tag{3.10.2}$$

and

$$S_1W + xV = S_1W + Z. \tag{3.10.3}$$

If $U = 0$ then $t = 0$ and hence $Y = W$ by definition, which shows (3.10.2) in this case. Now we turn to the case $U \neq 0$. We first argue that we may apply Lemma 3.9. Recall that the element $g_1 \in B = R[T_1,T_2,T_3]$ is of the form $T_1f_1 + T_2f_2 + T_3f_3$, where f_1, f_2, f_3 are the entries of the first column of φ. These entries generate an R-ideal of height 2 because $I = I_2(\varphi)$ has height two, and therefore $g_1 = T_1f_1 + T_2f_2 + T_3f_3$ is a prime element of B according to [14, Theorem]. Thus $B/(g_1)$ is a domain. Since $[\mathrm{Sym}(I)]_{(m,\_)} = [B/(g_1)]_{(m,\_)}$ for $m \le d_2 - 1$ and since $i \le d_2 - 1$, we deduce that multiplication by any linear form in R induces an injective S-linear map from $[\mathrm{Sym}(I)]_{(i-1,\_)}$ to $[\mathrm{Sym}(I)]_{(i,\_)}$. Now Lemma 3.9 shows that $\dim_k R_1U \ge \dim_k U + 1$ since $U \neq 0$. On the other hand, we have $R_1U \subset \mathcal{A}_{(i,j)} = W$, $\dim_k U = t$, and $\dim_k W = t+1$. Thus Claim (3.10.2) has been proved.

To show (3.10.3) we suppose that the containment $S_1W + xV \subset S_1W + Z$ is strict. In this case there exists a proper k-subspace Z' of Z, so that

$$S_1W + xV \subset S_1W + Z'.$$

In particular, $x(S_1U + V) \subset S_1W + Z'$ and then

$$x(SU + SV) \subset SW + SZ'. \tag{3.10.4}$$

However, the first S-module is isomorphic to $\mathcal{A}_{(i-1,\_)}$ and thus has rank d_1, whereas the number of generators of the second S-module is at most

$$\dim_k W + \dim_k Z' \le \dim_k W + \dim_k Z - 1 = d_1 - 1$$

and hence this module has rank at most $d_1 - 1$. This is a contradiction to the inclusion (3.10.4). □

4. Morley forms.

Adopt Data (2.1). The multiplication map

$$\mathcal{A}_i \otimes_S \mathrm{Sym}(I)_{\delta-i} \to \mathcal{A}_\delta \simeq S(-2)$$

is a perfect pairing of free S-modules. This perfect pairing is the starting point for the entire theory. It was first established by Jouanolou [21, 20]. We found Busé's description [4] of Jouanolou's

work to be very helpful. Our proof of this perfect pairing is given in Theorem 2.13. We have already highlighted this perfect pairing in (1.0.2) and (2.3.1). The above perfect pairing induces an isomorphism

$$\mathcal{A}_i \longrightarrow \operatorname{Hom}_S(\operatorname{Sym}(I)_{\delta-i}, S(-2)). \tag{4.0.5}$$

There are situations where we are able to identify an explicit basis for $\operatorname{Hom}_S(\operatorname{Sym}(I)_{\delta-i}, S(-2))$; see, for example, Lemmas 5.7 and 5.10. Our proof of Theorem 2.13 is highly non-constructive. On the other hand, Jouanolou's proof is constructive. He uses Morley forms to exhibit an explicit inverse for the isomorphism (4.0.5). We summarize Jouanolou's theory of Morley forms in the present section, and then we apply these ideas in Theorem 5.11 to exhibit an explicit generating set for $\mathcal{A}$ when, in the language of Data 2.1, $2 = d_1 < d_2$ and φ has a generalized zero in the first column.

Most of the present section is purely expository. We include this material for the reader's convenience and in order to put the ideas of Morley forms into the ambient notation. As far as we know, the calculation of "$q_{\beta,\delta-i-\beta}$" in part (5) of Observation 4.5 does not appear elsewhere in the literature; on the other hand, this calculation is straightforward. These "$q_{\beta,\delta-i-\beta}$" are the ingredient from the present section that is used in the proof of Theorem 5.11; see Corollary 4.4. We have calculated the "$q_{\beta,\delta-i-\beta}$" in greater generality than we use in the present paper because the calculation of these "$q_{\beta,\delta-i-\beta}$" is not the obstruction to generalizing Theorem 5.11; the obstruction is finding the appropriate generalization of Lemmas 5.7 and 5.10.

Begin with Data 2.1 and consider the ring $B\otimes_S B$. It is clear that the elements $g_j\otimes 1-1\otimes g_j$ of $B\otimes_S B$ are in the ideal $(x\otimes 1-1\otimes x, y\otimes 1-1\otimes y)$ for j equal to 1 and 2. Let H be a 2×2 matrix with entries in $B\otimes_S B$ so that

$$[g_1\otimes 1-1\otimes g_1 \quad g_2\otimes 1-1\otimes g_2] = [x\otimes 1-1\otimes x \quad y\otimes 1-1\otimes y]H. \tag{4.0.6}$$

Define Δ to be the image of $\det H$ in $\operatorname{Sym}(I)\otimes_S \operatorname{Sym}(I)$ under the natural map

$$B\otimes_S B \xrightarrow{\rho\otimes\rho} \operatorname{Sym}(I)\otimes_S \operatorname{Sym}(I),$$

where ρ is the natural quotient map

$$B\to B/(g_1,g_2) = \operatorname{Sym}(I).$$

Notice that Δ is bi-homogeneous of bi-degree $(\delta,2)$ in $\operatorname{Sym}(I)\otimes_S \operatorname{Sym}(I)$, where $x\otimes 1$, $1\otimes x$, $y\otimes 1$, and $1\otimes y$ all have bi-degree $(1,0)$ and T_1,T_2, and T_3 all have bi-degree $(0,1)$. One may also view $\operatorname{Sym}(I)\otimes_S \operatorname{Sym}(I)$ as having three degrees: $x\otimes 1$ and $y\otimes 1$ have tri-degree $(1,0,0)$, $1\otimes x$ and $1\otimes y$ have tri-degree $(0,1,0)$ and T_1,T_2, and T_3 all have tri-degree $(0,0,1)$. Write

$$\Delta = \sum_{i=0}^{\delta} \operatorname{morl}_{(i,\delta-i)},$$

where each

$$\operatorname{morl}_{(i,\delta-i)} \in (\operatorname{Sym}(I)\otimes_S \operatorname{Sym}(I))_{(i,\delta-i,2)}.$$

The tri-homogeneous elements

$$\{\mathrm{morl}_{(i,\delta-i)} \mid 0 \le i \le \delta\} \tag{4.0.7}$$

are the *Morley forms associated to the regular sequence* g_1, g_2 *in* B.

Observation 4.1. *Adopt* 2.1 *and let* syl *be a fixed generator for the one-dimensional vector space* $\mathcal{A}_{(\delta,2)}$, *as described in Remark* 2.17. *Then the following statements hold*:

1. $\mathrm{morl}_{(\delta,0)} = \alpha_1 \cdot \mathrm{syl} \otimes 1 \in \mathrm{Sym}(I)_\delta \otimes_S \mathrm{Sym}(I)_0$, *for some unit* α_1 *in* k,
2. $\mathrm{morl}_{(0,\delta)} = 1 \otimes \alpha_2 \cdot \mathrm{syl} \in \mathrm{Sym}(I)_0 \otimes_S \mathrm{Sym}(I)_\delta$, *for some unit* α_2 *in* k,
3. *if* L *is an element of the ideal* $(x,y)B$, *then*
$$(L \otimes 1 - 1 \otimes L)\Delta = 0$$
in $\mathrm{Sym}(I) \otimes_S \mathrm{Sym}(I)$, *and*
4. *if* b *is an element of the* S*-module* $\mathrm{Sym}(I)_\ell$, *for some non-negative integer* ℓ, *then*
$$(b \otimes 1)\,\mathrm{morl}_{(i,\delta-i)} = (1 \otimes b)\,\mathrm{morl}_{(i+\ell,\delta-i-\ell)}$$
in $\mathrm{Sym}(I)_{i+\ell} \otimes_S \mathrm{Sym}(I)_{\delta-i}$.

Proof. To prove (1) observe that $\mathrm{morl}_{(\delta,0)}$ is equal to the image of Δ under the the natural ring surjection

$$\mathrm{Sym}(I) \otimes_S \mathrm{Sym}(I) \to \mathrm{Sym}(I) \otimes_S \frac{\mathrm{Sym}(I)}{\mathfrak{m}\mathrm{Sym}(I)} = \mathrm{Sym}(I) \otimes_S S = \mathrm{Sym}(I).$$

On the other hand, the image of (4.0.6) in $B \otimes_S B/(B \otimes_S \mathfrak{m}B) = B$ is $[g_1, g_2] = [x,y]\overline{H}$, where $\overline{H}$ denotes the image of H and after permuting the rows of $\overline{H}$ this becomes the equation that is used, in Remark 2.17, to define syl.

The proof of (2) is completely analogous to the proof of (1). One sets $x \otimes 1$ and $y \otimes 1$ to zero instead of setting $1 \otimes x$ and $1 \otimes y$ to zero.

To show (3), multiply both sides of (4.0.6) on the right by the classical adjoint of H to see that the ideal

$$(x \otimes 1 - 1 \otimes x, y \otimes 1 - 1 \otimes y)\det H$$

of $B \otimes_S B$ is contained in the ideal $(g_1 \otimes 1 - 1 \otimes g_1, g_2 \otimes 1 - 1 \otimes g_2)$. The second ideal is sent to zero under the homomorphism $\rho \otimes \rho : B \otimes B \to \mathrm{Sym}(I) \otimes \mathrm{Sym}(I)$.

We now prove part (4). If $\ell = 0$, then b is in S and there is nothing to show. If ℓ is positive, then assertion (3) guarantees that $(b \otimes 1 - 1 \otimes b)\Delta = 0$. One completes the proof by examining the component of $(b \otimes 1 - 1 \otimes b)\Delta = 0$ in $\mathrm{Sym}(I)_{i+\ell} \otimes_S \mathrm{Sym}(I)_{\delta-i}$. $\square$

Now that the Morley forms have been defined, we set up the rest of the notation that is used in the statement of Theorem 4.2, where we establish that the Morley forms provide an inverse to the isomorphism of (4.0.5). If $u \in \mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i}, S)$, then u induces a map

$$\mathrm{Sym}(I) \otimes_S \mathrm{Sym}(I)_{\delta-i} \xrightarrow{1 \otimes u} \mathrm{Sym}(I) \otimes_S S = \mathrm{Sym}(I). \tag{4.1.1}$$

When this map is applied to $\mathrm{morl}_{(i,\delta-i)} \in \mathrm{Sym}(I)_i \otimes_S \mathrm{Sym}(I)_{\delta-i}$, the result is

$$(1\otimes u)(\mathrm{morl}_{(i,\delta-i)}) \in \mathrm{Sym}(I)_i.$$

It is shown in the proof of Theorem 4.2 that

$$(1\otimes u)(\mathrm{morl}_{(i,\delta-i)}) \text{ actually is in } \mathcal{A}_i. \tag{4.1.2}$$

Once (4.1.2) has been established, then it makes sense to define the S-module homomorphism

$$\nu_1 : \mathrm{Hom}_S(\mathrm{Sym}_{\delta-i}(I),S) \to \mathcal{A}_i \tag{4.1.3}$$

by

$$\nu_1(u) = (1\otimes u)(\mathrm{morl}_{(i,\delta-i)}).$$

We also define the S-module homomorphism

$$\nu_2 : \mathcal{A}_i \to \mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i},S). \tag{4.1.4}$$

If a is in $\mathcal{A}_i$, then multiplication by a is an S-module homomorphism

$$\mu_a : \mathrm{Sym}(I)_{\delta-i} \to \mathcal{A}_\delta.$$

It is well-known that A_δ is the free S-module generated by any fixed Sylvester element syl. (Our proof of this statement may be found in Remark 2.17.) Let $\mu_{\mathrm{syl}}^{-1} : \mathcal{A}_\delta \to S$ be the inverse of the isomorphism $\mu_{\mathrm{syl}} : S \to \mathcal{A}_\delta$. (The notation is consistent because $\mathrm{Sym}(I)_0 = S$.) In Remark 2.17, the homomorphism μ_{syl}^{-1} is called σ. If a is in $\mathcal{A}_i$, then we define $\nu_2(a)$ to be the homomorphism in $\mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i},S)$ which is given by

$$\mathrm{Sym}(I)_{\delta-i} \xrightarrow{\mu_a} \mathcal{A}_\delta \xrightarrow{\mu_{\mathrm{syl}}^{-1}} S.$$

Theorem 4.2. (**Jouanolou** [20, §3.6] *and* [21, §3.11]) *Adopt Data* 2.1 *and let i be an integer with* $0 \le i \le \delta$.

(1) *If $u \in \mathrm{Hom}(\mathrm{Sym}(I)_{\delta-i},S)$, then $(1\otimes u)(\mathrm{morl}_{(i,\delta-i)}) \in \mathcal{A}_i$.*
(2) *The homomorphisms*

$$\nu_2 : \mathcal{A}_i \to \mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i},S) \quad \textit{and} \quad \nu_1 : \mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i},S) \to \mathcal{A}_i,$$

as described in (4.1.4) *and* (4.1.3), *respectively, are inverses of one another* (*up to multiplication by a unit of k*).

Proof. Let $\mathrm{syl} \in \mathcal{A}_{\delta,2}$ be a fixed Sylvester form and α_1 and α_2 be the fixed units in k with

$$\mathrm{morl}_{(\delta,0)} = \alpha_1 \cdot \mathrm{syl} \otimes 1 \quad \text{and} \quad \mathrm{morl}_{(0,\delta)} = 1 \otimes \alpha_2 \cdot \mathrm{syl},$$

as described in parts (1) and (2) of Observation 4.1. Apply parts (4) and (1) of Observation 4.1 to see that

$$(b\otimes 1)\,\mathrm{morl}_{(i,\delta-i)} = (1\otimes b)\,\mathrm{morl}_{(\delta,0)} = \alpha_1 \cdot \mathrm{syl} \otimes b \in \mathrm{Sym}(I)_\delta \otimes_S \mathrm{Sym}(I)_{\delta-i}, \tag{4.2.1}$$

for all b in the S-module $\mathrm{Sym}(I)_{\delta-i}$.

Let u be an arbitrary element of $\mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i},S)$. Apply

$$1\otimes u : \mathrm{Sym}(I)\otimes_S \mathrm{Sym}(I)_{\delta-i}\to \mathrm{Sym}(I),$$

as described in (4.1.1), to each side of (4.2.1) to obtain

$$b\cdot(1\otimes u)(\mathrm{morl}_{(i,\delta-i)})=\alpha_1\cdot \mathrm{syl}\cdot u(b)\in \mathcal{A}_\delta.$$

Recall, from assertion (2) of Corollary 2.5, that $\mathcal{A}_{\delta+1}=0$. Thus, $\mathfrak{m}^{\delta-i+1}(1\otimes u)(\mathrm{morl}_{(i,\delta-i)})=0$; and therefore,

$$(1\otimes u)(\mathrm{morl}_{(i,\delta-i)})\in 0:_{\mathrm{Sym}(I)}\mathfrak{m}^{\delta-i+1}\subset 0:_{\mathrm{Sym}(I)}\mathfrak{m}^{\infty}=\mathcal{A};$$

see Remark 2.2. We have established assertion (1). We have also established half of assertion (2) because we have shown that $(\nu_2\circ\nu_1)(u)$ sends the element b of $\mathrm{Sym}(I)_{\delta-i}$ to

$$\mu_{\mathrm{syl}}^{-1}(b\cdot \nu_1(u))=\mu_{\mathrm{syl}}^{-1}(b\cdot(1\otimes u)(\mathrm{morl}_{(i,\delta-i)}))=\alpha_1\cdot\mu_{\mathrm{syl}}^{-1}(\mathrm{syl}\cdot u(b))=\alpha_1\cdot u(b);$$

and therefore, $\nu_2\circ\nu_1$ is equal to multiplication by the unit α_1 on $\mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i},S)$.

Now we prove the rest of (2). Let $a\in\mathcal{A}_i$. We compute

$$\begin{aligned}
(\nu_1\circ\nu_2)(a)&=(1\otimes\nu_2(a))(\mathrm{morl}_{(i,\delta-i)})\\
&=(1\otimes\mu_{\mathrm{syl}}^{-1}\circ\mu_a)(\mathrm{morl}_{(i,\delta-i)})\\
&=((1\otimes\mu_{\mathrm{syl}}^{-1})\circ(1\otimes\mu_a))(\mathrm{morl}_{(i,\delta-i)})\\
&=(1\otimes\mu_{\mathrm{syl}}^{-1})((1\otimes a)\,\mathrm{morl}_{(i,\delta-i)})\\
&=(1\otimes\mu_{\mathrm{syl}}^{-1})((a\otimes 1)\,\mathrm{morl}_{(0,\delta)}), &&\text{by Observation 4.1, part (4),}\\
&=(1\otimes\mu_{\mathrm{syl}}^{-1})(a\otimes\alpha_2\cdot\mathrm{syl})=\alpha_2\cdot a, &&\text{by Observation 4.1, part (2).}
\end{aligned}$$

□

Remark 4.3. Adopt Data 2.1 and let i be an integer with $0\le i\le\delta$. As an S-module $\mathrm{Sym}(I)_{\delta-i}$ is minimally generated by the monomials $x^\beta y^{\delta-i-\beta}$, with $0\le\beta\le\delta-i$. Thus, there are elements $q_{\beta,\delta-i-\beta}$ in $\mathrm{Sym}(I)_i$, with $0\le\beta\le\delta-i$, so that

$$\mathrm{morl}_{(i,\delta-i)}=\sum_{\beta=0}^{\delta-i}q_{\beta,\delta-i-\beta}\otimes x^\beta y^{\delta-i-\beta}\quad\text{in }\mathrm{Sym}(I)_i\otimes_S\mathrm{Sym}(I)_{\delta-i}.$$

Furthermore, if $u\in\mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i},S)$, then

$$\nu_1(u)=(1\otimes u)(\mathrm{morl}_{(i,\delta-i)})=\sum_{\beta=0}^{\delta-i}q_{\beta,\delta-i-\beta}\cdot u(x^\beta y^{\delta-i-\beta}).$$

Corollary 4.4. *Retain the notation and hypotheses of Remark* 4.3 *with* $d_1-1\le i\le d_2-2$. *Recall the exact sequence*

$$0\to\mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i},S)\to S^{\delta-i+1}\xrightarrow{\Upsilon_{d_2-i-1,1}{}^{\mathrm{T}}}S(1)^{d_2-i-1}\tag{4.4.1}$$

from part (2) *of Theorem* 2.7. *If* χ *is an element of* $S^{\delta-i+1}$ *with* $\Upsilon^{\mathrm{T}}_{d_2-i-1,1}\cdot\chi=0$, *then* χ *represents an element* u_χ *of* $\mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i},S)$ *and*

$$\nu_1(u_\chi)=\begin{bmatrix}q_{0,\delta-i} & \dots & q_{\delta-i,0}\end{bmatrix}\cdot\chi.$$

Proof. From part (2) of Theorem 2.7 we have an exact sequence

$$0 \to S^{d_2-i-1}(-1) \xrightarrow{\Upsilon_{d_2-i-1,1}} S^{\delta-i+1} \longrightarrow \operatorname{Sym}(I)_{\delta-i} \to 0,$$

where $S^{\delta-i+1} = B_{(\delta-i,\_)}$ is considered with the S-basis $y^{\delta-i}, \cdots, x^{\delta-i}$. Apply $\operatorname{Hom}_S(\_,S)$ to this sequence and use Remark 4.3. $\square$

An explicit formula for each $q_{\beta,\delta-i-\beta}$ is given in part (5) of the following Observation.

Observation 4.5. *Adopt Data* 2.1. *The following statements hold*: (1) – (4) *take place in* $B\otimes_S B$ *and* (5) *takes place in* $\operatorname{Sym}(I)\otimes_S \operatorname{Sym}(I)$.

(1) *If a and b are non-negative integers, then*

$$x^ay^b\otimes 1-1\otimes x^ay^b = (x\otimes 1-1\otimes x)\sum_{\beta=0}^{a-1} x^{a-1-\beta}\otimes x^\beta y^b + (y\otimes 1-1\otimes y)\sum_{\gamma=0}^{b-1} x^a y^{b-1-\gamma}\otimes y^\gamma.$$

(2) *If $g=\sum\limits_{\ell=0}^{d} c_\ell x^\ell y^{d-\ell}$ is an element of B, with each c_ℓ in S, then $g\otimes 1-1\otimes g$ is equal to*

$$(x\otimes 1-1\otimes x)\left(\sum_{\ell=0}^{d}\sum_{\beta=0}^{\ell-1} c_\ell x^{\ell-1-\beta}\otimes x^\beta y^{d-\ell}\right) + (y\otimes 1-1\otimes y)\left(\sum_{\lambda=0}^{d}\sum_{\gamma=0}^{d-\lambda-1} c_\lambda x^\lambda y^{d-\lambda-1-\gamma}\otimes y^\gamma\right).$$

(3) *If $g_1=\sum\limits_{\ell=0}^{d_1} c_{\ell,1}x^\ell y^{d_1-\ell}$ and $g_2=\sum\limits_{r=0}^{d_2} c_{r,2}x^r y^{d_2-r}$, then*

$$\begin{bmatrix} g_1\otimes 1-1\otimes g_1 & g_2\otimes 1-1\otimes g_2 \end{bmatrix} = \begin{bmatrix} (x\otimes 1-1\otimes x) & (y\otimes 1-1\otimes y) \end{bmatrix} H,$$

for

$$H = \begin{bmatrix} \sum\limits_{\ell=0}^{d_1}\sum\limits_{\beta=0}^{\ell-1} c_{\ell,1}x^{\ell-1-\beta}\otimes x^\beta y^{d_1-\ell} & \sum\limits_{\ell=0}^{d_2}\sum\limits_{\beta=0}^{\ell-1} c_{\ell,2}x^{\ell-1-\beta}\otimes x^\beta y^{d_2-\ell} \\ \sum\limits_{\lambda=0}^{d_1}\sum\limits_{\gamma=0}^{d_1-\lambda-1} c_{\lambda,1}x^\lambda y^{d_1-\lambda-1-\gamma}\otimes y^\gamma & \sum\limits_{\lambda=0}^{d_2}\sum\limits_{\gamma=0}^{d_2-\lambda-1} c_{\lambda,2}x^\lambda y^{d_2-\lambda-1-\gamma}\otimes y^\gamma \end{bmatrix}.$$

(4) *If H is the matrix of* (3), *then the determinant of H is equal to*

$$\sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}\sum_{w=0}^{i}\left[\sum_{(\ell,m)\in\mathfrak{S}_1} c_{\ell,1}c_{m,2} - \sum_{(\ell,m)\in\mathfrak{S}_2} c_{m,1}c_{\ell,2}\right] x^w y^{i-w}\otimes x^\beta y^{\delta-i-\beta},$$

where $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the following sets of pairs of non-negative integers:

$$\text{(4.5.1)}\qquad \mathfrak{S}_1 = \left\{(\ell,m)\;\middle|\; \begin{array}{l} \ell+m=w+1+\beta, \\ \beta+1\le\ell\le d_1, \text{ and} \\ 0\le m\le d_2-i-1+w \end{array}\right\} \quad \textit{and} \quad \mathfrak{S}_2 = \left\{(\ell,m)\;\middle|\; \begin{array}{l} \ell+m=w+1+\beta, \\ \beta+1\le\ell\le d_2, \text{ and} \\ 0\le m\le d_1-i-1+w \end{array}\right\}.$$

(5) *In the language of Remark* 4.3, *once i is fixed with $0\le i\le\delta$, then the Morley form* $\operatorname{morl}_{(i,\delta-i)}$ *is equal to* $\sum\limits_{\beta=0}^{\delta-i} q_{\beta,\delta-i-\beta}\otimes x^\beta y^{\delta-i-\beta}$ *in* $\operatorname{Sym}(I)_i\otimes_S\operatorname{Sym}(I)_{\delta-i}$ *with*

$$\text{(4.5.2)}\qquad q_{\beta,\delta-i-\beta} = \sum_{w=0}^{i}\left[\sum_{(\ell,m)\in\mathfrak{S}_1} c_{\ell,1}c_{m,2} - \sum_{(\ell,m)\in\mathfrak{S}_2} c_{m,1}c_{\ell,2}\right] x^w y^{i-w} \in \operatorname{Sym}(I)_i,$$

where $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the sets of (4.5.1).

Proof. Assertions (1) – (3) are straightforward calculations. We prove (4). Once (4) is established, then (5) follows immediately because the image of $\det H$ in $\operatorname{Sym}(I)\otimes_S \operatorname{Sym}(I)$ is equal to

$$\sum_{i=0}^{\delta}\operatorname{morl}_{(i,\delta-i)}=\sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}q_{\beta,\delta-i-\beta}\otimes x^{\beta}y^{\delta-i-\beta}.$$

We calculate $\det H = A-B$ with

$$\begin{aligned}
A &= \left(\sum_{\ell=0}^{d_1}\sum_{\beta=0}^{\ell-1}c_{\ell,1}x^{\ell-1-\beta}\otimes x^{\beta}y^{d_1-\ell}\right)\left(\sum_{\lambda=0}^{d_2}\sum_{\gamma=0}^{d_2-\lambda-1}c_{\lambda,2}x^{\lambda}y^{d_2-\lambda-1-\gamma}\otimes y^{\gamma}\right)\\
B &= \left(\sum_{\lambda=0}^{d_1}\sum_{\gamma=0}^{d_1-\lambda-1}c_{\lambda,1}x^{\lambda}y^{d_1-\lambda-1-\gamma}\otimes y^{\gamma}\right)\left(\sum_{\ell=0}^{d_2}\sum_{\beta=0}^{\ell-1}c_{\ell,2}x^{\ell-1-\beta}\otimes x^{\beta}y^{d_2-\ell}\right).
\end{aligned}$$

We put the summation signs that involve β on the left in order to see that

$$\begin{aligned}
A &= \sum_{\beta=0}^{d_1-1}\sum_{\ell=\beta+1}^{d_1}\sum_{\lambda=0}^{d_2}\sum_{\gamma=0}^{d_2-\lambda-1} c_{\ell,1}c_{\lambda,2}x^{\ell-1-\beta+\lambda}y^{d_2-\lambda-1-\gamma}\otimes x^{\beta}y^{d_1-\ell+\gamma}\\
B &= \sum_{\beta=0}^{d_2-1}\sum_{\ell=\beta+1}^{d_2}\sum_{\lambda=0}^{d_1}\sum_{\gamma=0}^{d_1-\lambda-1} c_{\lambda,1}c_{\ell,2}x^{\ell-1-\beta+\lambda}y^{d_1-\lambda-1-\gamma}\otimes x^{\beta}y^{d_2-\ell+\gamma}.
\end{aligned}$$

Replace γ with $d_2-2-i-\beta+\ell$ in A and with $d_1-2-i-\beta+\ell$ in B to obtain

$$\begin{aligned}
A &= \sum_{\beta=0}^{d_1-1}\sum_{\ell=\beta+1}^{d_1}\sum_{\lambda=0}^{d_2}\sum_{i=\ell-1-\beta+\lambda}^{d_2-2-\beta+\ell} c_{\ell,1}c_{\lambda,2}x^{\ell-1-\beta+\lambda}y^{\beta+i+1-\ell-\lambda}\otimes x^{\beta}y^{\delta-i-\beta}\\
B &= \sum_{\beta=0}^{d_2-1}\sum_{\ell=\beta+1}^{d_2}\sum_{\lambda=0}^{d_1}\sum_{i=\ell-1-\beta+\lambda}^{d_1-2-\beta+\ell} c_{\lambda,1}c_{\ell,2}x^{\ell-1-\beta+\lambda}y^{\beta+i+1-\ell-\lambda}\otimes x^{\beta}y^{\delta-i-\beta}.
\end{aligned}$$

We re-arrange the order of summation by putting the sum involving i first. We see that i satisfies:

$$0\le \ell-1-\beta+\lambda\le i\le d_2-2-\beta+\ell\le\delta-\beta\le\delta$$

in A. The analogous inequalities hold in B; in particular, i also satisfies $0\le i\le\delta$. The old constraints on i now become constraints on λ and ℓ. It follows that

$$\begin{aligned}
A &= \sum_{i=0}^{\delta}\sum_{\beta=0}^{d_1-1}\sum_{\ell=\max\{\beta+1,i-d_2+2+\beta\}}^{d_1}\sum_{\lambda=0}^{\min\{d_2,\beta+i+1-\ell\}} c_{\ell,1}c_{\lambda,2}x^{\ell-1-\beta+\lambda}y^{\beta+i+1-\ell-\lambda}\otimes x^{\beta}y^{\delta-i-\beta}\\
B &= \sum_{i=0}^{\delta}\sum_{\beta=0}^{d_2-1}\sum_{\ell=\max\{\beta+1,i-d_1+2+\beta\}}^{d_2}\sum_{\lambda=0}^{\min\{d_1,\beta+i+1-\ell\}} c_{\lambda,1}c_{\ell,2}x^{\ell-1-\beta+\lambda}y^{\beta+i+1-\ell-\lambda}\otimes x^{\beta}y^{\delta-i-\beta}.
\end{aligned}$$

In A, the third summation sign represents the empty sum unless $\beta\le d_1-1$ and $\beta\le\delta-i$. Thus, the following four choices for the second summation sign all yield the same value for A:

$$\sum_{0\le\beta},\quad \sum_{\beta=0}^{d_1-1},\quad \sum_{\beta=0}^{\delta-i},\quad \text{or}\quad \sum_{\beta=0}^{\min\{d_1-1,\delta-i\}}.$$

An analogous statement holds for B. We conclude that

$$\begin{aligned}
A &= \sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}\sum_{\ell=\max\{\beta+1,i-d_2+2+\beta\}}^{d_1}\sum_{\lambda=0}^{\min\{d_2,\beta+i+1-\ell\}} c_{\ell,1}c_{\lambda,2}x^{\ell-1-\beta+\lambda}y^{\beta+i+1-\ell-\lambda}\otimes x^{\beta}y^{\delta-i-\beta}\\
B &= \sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}\sum_{\ell=\max\{\beta+1,i-d_1+2+\beta\}}^{d_2}\sum_{\lambda=0}^{\min\{d_1,\beta+i+1-\ell\}} c_{\lambda,1}c_{\ell,2}x^{\ell-1-\beta+\lambda}y^{\beta+i+1-\ell-\lambda}\otimes x^{\beta}y^{\delta-i-\beta}.
\end{aligned}$$

Replace λ with $w-\ell+1+\beta$ to obtain

$$\begin{aligned}
A &= \sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}\sum_{\ell=\max\{\beta+1,i-d_2+2+\beta\}}^{d_1}\sum_{w=\ell-1-\beta}^{\min\{i,d_2+\ell-1-\beta\}} c_{\ell,1}c_{w-\ell+1+\beta,2}x^{w}y^{i-w}\otimes x^{\beta}y^{\delta-i-\beta}\\
B &= \sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}\sum_{\ell=\max\{\beta+1,i-d_1+2+\beta\}}^{d_2}\sum_{w=\ell-1-\beta}^{\min\{i,d_1+\ell-1-\beta\}} c_{w-\ell+1+\beta,1}c_{\ell,2}x^{w}y^{i-w}\otimes x^{\beta}y^{\delta-i-\beta}.
\end{aligned}$$

Exchange the third and fourth summation signs. Keep in mind that $0\le \ell-1-\beta\le w\le i$. We see that

$$\begin{aligned}
A &= \sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}\sum_{w=0}^{i}\sum_{\ell=\max\{\beta+1,i-d_2+2+\beta,w+1+\beta-d_2\}}^{\min\{d_1,w+1+\beta\}} c_{\ell,1}c_{w-\ell+1+\beta,2}x^{w}y^{i-w}\otimes x^{\beta}y^{\delta-i-\beta}\\
B &= \sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}\sum_{w=0}^{i}\sum_{\ell=\max\{\beta+1,i-d_1+2+\beta,w+1+\beta-d_1\}}^{\min\{d_2,w+1+\beta\}} c_{w-\ell+1+\beta,1}c_{\ell,2}x^{w}y^{i-w}\otimes x^{\beta}y^{\delta-i-\beta}.
\end{aligned}$$

The parameter w satisfies $0\le w\le i$; hence, $w+1+\beta-d_s\le i-d_s+2+\beta$, for s equal to 1 or 2, and

$$\begin{aligned}
A &= \sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}\sum_{w=0}^{i}\sum_{\ell=\max\{\beta+1,i-d_2+2+\beta\}}^{\min\{d_1,w+1+\beta\}} c_{\ell,1}c_{w-\ell+1+\beta,2}x^{w}y^{i-w}\otimes x^{\beta}y^{\delta-i-\beta}\\
B &= \sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}\sum_{w=0}^{i}\sum_{\ell=\max\{\beta+1,i-d_1+2+\beta\}}^{\min\{d_2,w+1+\beta\}} c_{w-\ell+1+\beta,1}c_{\ell,2}x^{w}y^{i-w}\otimes x^{\beta}y^{\delta-i-\beta}.
\end{aligned}$$

Let $m=w-\ell+1+\beta$. The four constraints

$$\beta+1\le\ell,\quad i-d_2+2+\beta\le\ell,\quad \ell\le d_1,\quad \ell\le w+\beta+1$$

on ℓ in A are equivalent to

$$\beta+1\le\ell,\quad m\le d_2+w-1-i,\quad \ell\le d_1,\quad 0\le m,$$

respectively; and the four constraints

$$\beta+1\le\ell,\quad i-d_1+2+\beta\le\ell,\quad \ell\le d_2,\quad \ell\le w+1+\beta$$

on ℓ in B are equivalent to

$$\beta+1\le\ell,\quad m\le d_1+w-i-1,\quad \ell\le d_2,\quad 0\le m,$$

respectively. It follows that

$$A=\sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}\sum_{w=0}^{i}\sum_{(\ell,m)\in\mathfrak{S}_1} c_{\ell,1}c_{m,2}x^{w}y^{i-w}\otimes x^{\beta}y^{\delta-i-\beta}$$

$$B=\sum_{i=0}^{\delta}\sum_{\beta=0}^{\delta-i}\sum_{w=0}^{i}\sum_{(\ell,m)\in\mathfrak{S}_2} c_{m,1}c_{\ell,2}x^{w}y^{i-w}\otimes x^{\beta}y^{\delta-i-\beta},$$

as desired. $\square$

The answer of part (5) of Observation 4.5 can be simplified when $d_1 = 2$ and $1 \le i \le d_2 - 1$. This hypothesis is in effect when we apply Observation 4.5 in the proof of Theorem 5.11. Once $d_1 = 2$, then δ is equal to d_2 and we use d_2 in place of δ in our simplification. If P is a statement, then define

$$\underline{\chi}(P) = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false}. \end{cases} \tag{4.5.3}$$

Corollary 4.6. *If $d_1 = 2$, $1 \le i \le d_2 - 1$, and $0 \le \beta \le d_2 - i$, then the element $q_{\beta,d_2-i-\beta}$ of (4.5.2) is equal to*

$$q_{\beta,d_2-i-\beta} = \begin{cases} \underline{\chi}(\beta = 0)\left(\sum\limits_{w=0}^{i} c_{1,1}c_{w,2}x^w y^{i-w} + \sum\limits_{w=1}^{i} c_{2,1}c_{w-1,2}x^w y^{i-w}\right) + \underline{\chi}(\beta = 1)\sum\limits_{w=0}^{i} c_{2,1}c_{w,2}x^w y^{i-w} \\ -\underline{\chi}(\beta \le d_2 - i - 1)c_{0,1}c_{i+1+\beta,2}x^i - c_{1,1}c_{i+\beta,2}x^i - c_{0,1}c_{i+\beta,2}x^{i-1}y. \end{cases}$$

Proof. The parameter d_1 is equal to 2; so $\delta = d_2$. Write $q_{\beta,\delta-i-\beta}$, from (4.5.2), as $A + B$, where

$$A = \sum_{w=0}^{i}\sum_{(\ell,m)\in\mathfrak{S}_1} c_{\ell,1}c_{m,2}x^w y^{i-w} \quad\text{and}\quad B = -\sum_{w=0}^{i}\sum_{(\ell,m)\in\mathfrak{S}_2} c_{m,1}c_{\ell,2}x^w y^{i-w}.$$

Since $d_1 = 2$, the constraint $\beta + 1 \le \ell \le d_1$ in the definition of $\mathfrak{S}_1$ allows only three possible values for the pair (β,ℓ); namely, (β,ℓ) is equal to $(0,1)$, or $(0,2)$, or $(1,2)$. For each of these pairs, one sets $m = w + 1 + \beta - \ell$ and then one verifies that $0 \le m \le d_2 - i - 1 + w$ becomes $1 \le w$ when $(\beta,\ell) = (0,2)$ and automatically holds otherwise. It follows that

$$A = \underline{\chi}(\beta = 0)\left(\sum_{w=0}^{i} c_{1,1}c_{w,2}x^w y^{i-w} + \sum_{w=1}^{i} c_{2,1}c_{w-1,2}x^w y^{i-w}\right) + \underline{\chi}(\beta = 1)\sum_{w=0}^{i} c_{2,1}c_{w,2}x^w y^{i-w}.$$

Now we simplify B. The constraint $0 \le m \le d_1 - i - 1 + w$ in the definition of $\mathfrak{S}_2$ becomes

$$0 \le m \le w + 1 - i,$$

when $d_1 = 2$. On the other hand, the parameter w in B is always at most i. Thus, the pair (w,m) must satisfy $0 \le m \le w + 1 - i$ and $w \le i \le w + 1$. It follows that there are only three possible values for the pair (w,m), namely, (w,m) is equal to $(i,0)$ or $(i,1)$, or $(i-1,0)$. Use $\ell + m = w + 1 + \beta$ to define ℓ. Verify that $\beta + 1 \le \ell \le d_2$ becomes $\beta \le d_2 - i - 1$ when $(w,m) = (i,0)$ and holds automatically otherwise. It follows that

$$B = -\underline{\chi}(\beta \le d_2 - i - 1)c_{0,1}c_{i+1+\beta,2}x^i - c_{1,1}c_{i+\beta,2}x^i - c_{0,1}c_{i+\beta,2}x^{i-1}y.$$

□

5. Explicit generators for $\mathcal{A}$ when $d_1 = 2$.

Adopt Data (2.1) with $d_1 = 2$. If $d_2 = 2$, then the generators of $\mathcal{A}_{\ge 1}$ are explicitly described in Corollary 2.16. If φ does not have a generalized zero in the first column, then Busé [4, Prop. 3.2] gave explicit formula for the generators of $\mathcal{A}_{\ge 1}$. The present section is concerned with the following situation.

T-deg											
d	1^*										
$\vdots$											
$\lceil \frac{d}{2} \rceil$		1^*									
$\lfloor \frac{d}{2} \rfloor$		1^*	2	1							
$\lfloor \frac{d}{2} \rfloor - 1$				1^*	$\cdots$						
$\vdots$											
4					$\cdots$	1					
3						1^*	2	1			
2								1^*	2	1	
	0	1	2	3	$\cdots$	d_2-4	d_2-3	d_2-2	d_2-1	d_2	xy-deg

Table 5.1.1. The generator degrees for the free S-module $\mathcal{A}$, in the presence of Data 5.1, when d is odd. (*The elements that correspond to the generator degrees marked by* $*$ *are minimal generators for the* $\operatorname{Sym}(I)$*-ideal* $\mathcal{A}$.)

Data 5.1. Adopt Data (2.1) with $2 = d_1 < d_2$. Assume also that φ has a generalized zero in the first column.

Let $\mathcal{C}$ be the curve of Remark 2.9. We recall that if the parameterization $\mathbb{P}^1 \to \mathcal{C}$ is birational, then the hypothesis that φ has a generalized zero in the first column is equivalent to the statement that there is a singularity of multiplicity d_2 on $\mathcal{C}$.

In this section we assume that Data 5.1 is in effect and we describe explicitly **all** of the defining equations of the Rees algebra $\mathcal{R}$. Of course, the results of Section 3 apply in the present section; so we know the degrees of the generators of $\mathcal{A}_{\geq d_1-1} = \mathcal{A}_{\geq 1}$, a priori, from Table 3.5.1. Indeed, in the context of the present section, Table 3.5.1 is given in Tables 5.1.1 and 5.1.2. There are two ways in which the present tables are simpler than the general table. First of all, the description of the generator degrees depends on the remainder of a division by d_1. When d_1 is two, there are only two possible remainders: 0 or 1; so, there are many fewer $\dots$'s in the present tables. Secondly, Table 3.5.1, together with Corollary 3.10, describes the degrees of the S-module, and the B-module, generators of $\mathcal{A}_{\geq d_1-1}$. When $d_1 = 2$, then $\mathcal{A}_{\geq d_1-1}$ is, in fact, equal to all of $\mathcal{A}_{\geq 1}$. At any rate, in the present section we give much more than the degrees of the generators. We give explicit formulas for the minimal generators of $\mathcal{A}_{\geq 1}$.

Remark 5.2. We notice, with significant interest, how similar Tables 5.1.1 and 5.1.2 are to the degree tables of [3, Rmk. 5.6]. It appears that the tables of [3] are the transpose of the tables given here. This observation is particularly striking because there is virtually no overlap between the data of [3] and the data used here. In the present section, $d_1 = 2$ and $d_2 = d-2$; so the two parameters d_1 and d_2 are almost as far apart as possible. On the other hand, in [3], the parameters are as close as possible: $d_1 = \lfloor \frac{d}{2} \rfloor$ and $d_2 = \lceil \frac{d}{2} \rceil$.

T-deg											
d	1^*										
$\vdots$											
$\frac{d}{2}$		2^*	1								
$\frac{d}{2}-1$			1^*	2	$\cdots$						
$\vdots$											
4					$\cdots$	1					
3						1^*	2	1			
2								1^*	2	1	
	0	1	2	3	$\cdots$	d_2-4	d_2-3	d_2-2	d_2-1	d_2	xy-deg

Table 5.1.2. The generator degrees for the free S-module $\mathcal{A}$, in the presence of Data 5.1, when d is even and and the morphism $\mathbb{P}^1 \to \mathcal{C}$ of Remark 2.9 is birational. (*The elements that correspond to the generator degrees marked by $*$ are minimal generators for the* $\operatorname{Sym}(I)$*-ideal $\mathcal{A}$.*)

We recall that the S-module $\mathcal{A}_0$ is free of rank 1 and is generated in degree $(0, \deg \mathcal{C})$ by the implicit equation $F(T_1,T_2,T_3)$ of the curve $\mathcal{C}$ of Remarks 2.9. Furthermore, F is an r^{th} root of the resultant of g_1 and g_2, where $r = 1$ if the parameterization $\mathbb{P}^1 \to \mathcal{C}$ of Remark 2.9 is birational, and $r = 2$ otherwise. Degree considerations show that, when the parameterization $\mathbb{P}^1 \to \mathcal{C}$ is birational, then F together with the minimal generators of the $\operatorname{Sym}(I)$-ideal $\mathcal{A}_{\geq 1}$, form a minimal generating set for the $\operatorname{Sym}(I)$-ideal $\mathcal{A}$. The parameterization $\mathbb{P}^1 \to \mathcal{C}$ is guaranteed to be birational when d_2 is odd; see [24] or [6, Thm. 0.10]. If the parameterization $\mathbb{P}^1 \to \mathcal{C}$ is not birational, then one can reparameterize in order to obtain a birational parameterization. The column degrees of the Hilbert-Burch matrix φ', which corresponds to the new parameterization, are $d_1' = 1$ and $d_2' = \frac{d_2}{2}$. The matrix φ' is "almost linear". The defining equations of the Rees algebra associated to φ' are recorded explicitly in [23, Sect. 3], as well as [7, Thm. 2.3] and [15, Prop. 4.3].

We first show how to modify arbitrary Data 5.1 into data in a canonical form.

Observation 5.3. *If Data* 5.1 *is adopted with k a field which is closed under the taking of square root, then one may assume that the first column of φ is either $[x^2+y^2, xy, 0]^{\mathrm{T}}$ or $[y^2, x^2, 0]^{\mathrm{T}}$.*

Proof. Let φ_1 represent the first column of φ. We may apply invertible row operations to φ and linear change of variables to $R = k[x,y]$ without changing the ideal I, the symmetric algebra $\operatorname{Sym}(I)$, the Rees algebra $\mathcal{R}$, or any other essential feature of Data 5.1. The hypothesis about the generalized zero allows us to use invertible row operations to put a zero into the bottom position of φ_1. The non-zero entries of φ_1 each factor into a product of linear forms. If both of these entries are perfect squares, then, after a linear change of variables, $\varphi_1 = [y^2, x^2, 0]^{\mathrm{T}}$. Otherwise, φ_1 can be put in the form $\varphi = [\alpha_1 x^2 + \alpha_2 y^2, xy, 0]^{\mathrm{T}}$, for some constants α_1 and α_2 in k. The hypothesis about the height of $I_1(\varphi)$ ensures that both α's are units in k. Another linear change of variables yields the result. $\square$

Corollary 5.4. *If Data* 5.1 *is adopted, i is an integer with $1 \le i \le d_2-2$, then the matrix $\Upsilon^{\mathrm{T}}_{d_2-i-1,1}$ of Corollary* 4.4 *is the $d_2-i-1 \times d_2-i+1$ matrix*

$$\begin{bmatrix} T_1 & T_2 & T_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & T_1 & T_2 & T_1 \end{bmatrix} \text{ if the first column of } \varphi \text{ is } [x^2+y^2, xy, 0]^{\mathrm{T}} \tag{5.4.1}$$

$$\begin{bmatrix} T_1 & 0 & T_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & T_1 & 0 & T_2 \end{bmatrix} \text{ if the first column of } \varphi \text{ is } [y^2, x^2, 0]^{\mathrm{T}}. \tag{5.4.2}$$

Proof. According to Definition 2.6, $\Upsilon^{\mathrm{T}}_{d_2-i-1,1}$ is the $d_2-i-1 \times d_2-i+1$ matrix

$$\begin{bmatrix} c_{0,1} & c_{1,1} & c_{2,1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & c_{0,1} & c_{1,1} & c_{2,1} \end{bmatrix},$$

where $g_1 = c_{0,1}y^2 + c_{1,1}xy + c_{2,1}x^2$. On the other hand,

$$g_1 = [T_1, T_2, T_3] \begin{bmatrix} x^2+y^2 \\ xy \\ 0 \end{bmatrix} = T_1(x^2+y^2) + T_2 xy \quad \text{or} \quad g_1 = [T_1, T_2, T_3] \begin{bmatrix} y^2 \\ x^2 \\ 0 \end{bmatrix} = T_1 y^2 + T_2 x^2.$$

□

Our intention is to apply the technique of Corollary 4.4 when the hypotheses of Observation 5.3 are in effect. For that reason, we next find the relations on matrices like those of (5.4.1) and (5.4.2). In the language of Definition 5.5, the matrix of (5.4.1) is A_ℓ with $\ell = d_2-i+1$ and the matrix of (5.4.2) is $\mathfrak{A}_\ell$ with $\ell = d_2-i+1$.

Definition 5.5. For each integer ℓ, with $3 \le \ell$, let A_ℓ and $\mathfrak{A}_\ell$ be the following $(\ell-2) \times \ell$ matrices with entries in the polynomial ring $U = k[T_1, T_2]$:

$$A_\ell = \begin{bmatrix} T_1 & T_2 & T_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & T_1 & T_2 & T_1 \end{bmatrix} \quad \text{and} \quad \mathfrak{A}_\ell = \begin{bmatrix} T_1 & 0 & T_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & T_1 & 0 & T_2 \end{bmatrix}.$$

In Lemmas 5.7 and 5.10 we resolve $\operatorname{coker} A_\ell$ and $\operatorname{coker} \mathfrak{A}_\ell$, respectively. In each case, the answer depends on the parity of ℓ. We decompose each A_ℓ into four pieces of approximately equal size. The relations on A_ℓ are constructed from maximal minors of these smaller matrices. In fact, up to

re-arrangement of the rows and columns, there are only two constituent pieces for the A_ℓ. We call the two primary constituent pieces B_k and $L_{a\times b}$.

(5.5.1) For each integer k, with $2 \le k$, let B_k be the $(k-1)\times k$ matrix A_{k+1} with the last column removed.

For example,

$$B_2 = \begin{bmatrix} T_1 & T_2 \end{bmatrix}, \quad \text{and} \quad B_3 = \begin{bmatrix} T_1 & T_2 & T_1 \\ 0 & T_1 & T_2 \end{bmatrix}.$$

For each pair of positive integers a,b, let $L_{a\times b}$ be the $a\times b$ matrix with T_1 in the lower left hand corner and zero everywhere else.

(5.5.2) If C is a matrix, then let let $C^\dagger$ represent the matrix that is obtained from C by re-arranging both the rows and the columns of C in the exact opposite order.

In other words, if J_k is the $k\times k$ matrix with

$$(J_k)_{i,j} = \begin{cases} 1 & \text{if } i+j=k+1 \\ 0 & \text{otherwise,} \end{cases}$$

and C is an $a\times b$ matrix, then

$$C^\dagger = J_a C J_b$$

For example,

$$B_3^\dagger = \begin{bmatrix} T_2 & T_1 & 0 \\ T_1 & T_2 & T_1 \end{bmatrix},$$

and the matrix $L^\dagger_{a\times b}$ is the $a\times b$ matrix with T_1 in the upper right hand corner and zero everywhere else. We find it mnemonically helpful to write $u_{a\times b} = L^\dagger_{a\times b}$.

Observe that

$$A_\ell = \begin{cases} \begin{bmatrix} B_k & L_{k-1\times k} \\ u_{k-1\times k} & B_k^\dagger \end{bmatrix} & \text{if } \ell = 2k \\ \\ \begin{bmatrix} B_k & L_{k-1\times k+1} \\ u_{k\times k} & B_{k+1}^\dagger \end{bmatrix} & \text{if } \ell = 2k+1. \end{cases}$$

We collect a few properties of the objects we have defined.

Observation 5.6. *The following statements hold.*

(1) *The operation* † *respects matrix multiplication in the sense that* $(AB)^\dagger = A^\dagger B^\dagger$.
(2) *The first column of* A_{k+1} *is* $[T_1, 0, \dots, 0]^{\mathrm{T}}$.
(3) *If the first column of* A_{k+1} *is deleted, then the resulting matrix is* $B_k^\dagger$.
(4) *The last column of* A_{k+1} *is* $[0, \dots, 0, T_1]^{\mathrm{T}}$.
(5) *If the last column of* A_{k+1} *is deleted, then the resulting matrix is* B_k.
(6) *If the last row of* B_k *is deleted, then the resulting matrix is* A_k.
(7) *If the top row of* $B_k^\dagger$ *is deleted, then the resulting matrix is* A_k.
(8) *If* $v = [v_1, \dots, v_k]^{\mathrm{T}}$ *is a vector and* $B_k^\dagger v = 0$, *then*

(a) *the matrix that is obtained from* $B_k^\dagger$ *by deleting the top row sends* $v' = [0, v_1, \dots, v_{k-1}]^{\mathrm{T}}$ *to zero, and*

(b) *the matrix that is obtained from* $B_{k+1}^\dagger$ *by deleting the top row sends* $v'' = [0, v]^{\mathrm{T}}$ *to zero.*

(9) *If* $v = [v_1, \dots, v_{k+1}]^{\mathrm{T}}$ *is a vector and* $B_{k+1}^\dagger v = 0$*, then the matrix that is obtained from* B_k *by deleting the bottom row sends* $v' = [v_{k-1}, \dots, v_1, 0]^{\mathrm{T}}$ *to zero.*

Proof. Assertions (1) – (7) are obvious. We prove (8) and (9). Start with $B_k^\dagger v = 0$. Apply (3) to see that

$$A_{k+1}\begin{bmatrix}0\\ v\end{bmatrix} = 0.$$

The only non-zero entry in the last column of A_{k+1} lives in the bottom row, see (4). It follows that A_{k+1}, with the last row and the last column deleted, sends v' to zero. Apply (5) and (6) to conclude $A_k v' = 0$. On the other hand, we saw in (7) that A_k is $B_k^\dagger$ with the top row removed, and the proof of (8.a) is complete. Now apply (3) to $B_k^\dagger v = 0$ to see that $A_{k+1}v'' = 0$. On the other hand, (7) shows that $B_{k+1}^\dagger$, with row 1 removed, is A_{k+1} and the proof of (8.b) is complete.

We prove (9). The hypothesis that $B_{k+1}^\dagger v = 0$ implies that the matrix that is obtained from $B_{k+1}^\dagger$ by deleting the last two rows sends v to zero. Observe that the only non-zero entries of the last two columns of $B_{k+1}^\dagger$ live in the bottom two rows. Thus, the matrix that is obtained from $B_{k+1}^\dagger$ by deleting the last two rows and the last two columns sends $v'' = [v_1, \dots, v_{k-1}]^{\mathrm{T}}$ to zero. On the other hand, the matrix that is obtained from $B_{k+1}^\dagger$ by deleting the last two rows and the last two columns is $B_{k-1}^\dagger$. Thus, $B_{k-1}^\dagger v'' = 0$.

Apply (7) and (3) to see that the matrix which is obtained from $B_k^\dagger$ by removing the first row and column is $B_{k-1}^\dagger$. Thus, the matrix which is obtained from $B_k^\dagger$ by removing the first row sends $v''' = [0, v_1, \dots, v_{k-1}]^{\mathrm{T}}$ to 0. Apply the operation † to the most recent statement. Assertion (1) guarantees that the matrix that is obtained from B_k by removing the bottom row sends $v'''^\dagger$ to zero. The proof of (9) is complete because $v'''^\dagger$ is equal to v'. □

Lemma 5.7. *Adopt the notation of Definition 5.5 and* (5.5.1)*.*

(1) *If* ℓ *is the even integer* $2k$*,* $m_1, \dots, m_k$ *are the signed maximal order minors of* B_k*, and*

$$C = \begin{bmatrix} m_1 & \dots & m_{k-1} & m_k & 0 & -m_k & \dots & -m_2 \\ -m_2 & \dots & -m_k & 0 & m_k & m_{k-1} & \dots & m_1 \end{bmatrix}^{\mathrm{T}},$$

then

$$0 \to U(-k)^2 \xrightarrow{C} U(-1)^\ell \xrightarrow{A_\ell} U^{\ell-2} \tag{5.7.1}$$

is exact.

(2) *If* ℓ *is the odd integer* $2k+1$*,* $m_1, \dots, m_k$ *are the signed maximal order minors of* B_k*,* $M_1, \dots, M_{k+1}$ *are the signed maximal order minors of* $B_{k+1}^\dagger$*, and*

$$C = \begin{cases} \begin{bmatrix} m_1 & \dots & m_{k-1} & m_k & 0 & -m_k & \dots & -m_1 \\ -M_{k-1} & \dots & -M_1 & 0 & M_1 & M_2 & \dots & M_{k+1} \end{bmatrix}^{\mathrm{T}} & \textit{for } 5 \le \ell \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & T_2 & -T_1 \end{bmatrix}^{\mathrm{T}} & \textit{for } 3 = \ell, \end{cases}$$

then

$$0 \to U(-k)\oplus U(-k-1) \xrightarrow{C} U(-1)^{\ell} \xrightarrow{A_\ell} U^{\ell-2} \tag{5.7.2}$$

is exact.

Proof. Denote the i^{th} column of each matrix C by C_i.

We prove (1). We first show that (5.7.1) is a complex. Separate the first column of C into two pieces:

$$C_1 = \begin{bmatrix} t \\ b \end{bmatrix} \quad \text{with} \quad t = \big[m_1,\dots,m_k\big]^{\mathrm{T}} \quad \text{and} \quad b = \big[0,-m_k,\dots,-m_2\big]^{\mathrm{T}}.$$

The definition of the m's gives $B_k t = 0$. The product $L_{k-1\times k}b$ is also zero because the only non-zero entry in $L_{k-1\times k}$ is multiplied by zero. The product of row k from A_ℓ times C_1 is $T_1m_k - T_1m_k = 0$. Apply (1) from Observation 5.6 to the equation $B_kt = 0$ to see that $B_k^\dagger$ sends $t^\dagger = [m_k,\dots,m_1]^{\mathrm{T}}$ to zero. It follows from (8.a) in Observation 5.6 that the bottom $k-2$ rows of $B_k^\dagger$ kills b. (The minus signs do not cause any difficulty.) We have shown that $A_\ell C_1 = 0$. It follows from (1) of Observation 5.6 that $A_\ell^\dagger C_1^\dagger = 0$; but $A_\ell^\dagger = A_\ell$ and $C_1^\dagger = C_2$. It follows that $A_\ell C_2 = 0$ and therefore $A_\ell C = 0$ and (5.7.1) is a complex. We apply the Buchsbaum-Eisenbud criteria to show that (5.7.1) is exact. The ideal of 2×2 minors of C has grade two since

$$\begin{vmatrix} m_k & 0 \\ 0 & m_k \end{vmatrix} = \pm T_1^{\ell-2} \quad \text{and} \quad \begin{vmatrix} m_1 & -m_2 \\ -m_2 & m_1 \end{vmatrix} \equiv \pm T_2^{\ell-2} \quad \mathrm{mod}\ (T_1).$$

Assertion (1) has been established.

Assertion (2) is obvious when $\ell = 3$. Henceforth, we assume $5 \le \ell$. We next show that (5.7.2) is a complex. Write

$$C_1 = \begin{bmatrix} t \\ b \end{bmatrix} \quad \text{with} \quad t = \big[m_1,\dots,m_k\big]^{\mathrm{T}} \quad \text{and} \quad b = \big[0,-m_k,\dots,-m_1\big]^{\mathrm{T}}.$$

The definition of the m's ensures that $B_kt = 0$. The product $L_{k-1\times k+1}b$ is zero because the only non-zero entry of $L_{k-1\times k+1}$ is multiplied by zero. The product of row k of A_ℓ times C_1 is $T_1m_k - T_1m_k = 0$. The bottom $k-1$ rows of $u_{k\times k}$ are identically zero. Apply (8.b) from Observation 5.6 to $B_k^\dagger t^\dagger = 0$ in order to see that the bottom $k-1$ rows of $B_{k+1}^\dagger$ times b is equal to zero. (Again, the signs play no role in this part of the calculation.) We have shown that $A_\ell C_1 = 0$. Write

$$C_2 = \begin{bmatrix} t \\ b \end{bmatrix} \quad \text{with} \quad t = \big[-M_{k-1},\dots,-M_1,0\big]^{\mathrm{T}} \quad \text{and} \quad b = \big[M_1,\dots,M_{k+1}\big]^{\mathrm{T}}.$$

The definition of the M's guarantees that $B_{k+1}^\dagger b = 0$. The product $u_{k\times k}t$ is zero because the only non-zero entry of $u_{k\times k}$ is multiplied by zero. The product of row $k-1$ of A_ℓ times C_2 is $-T_1M_1 + T_1M_1 = 0$. The top $k-2$ rows of $L_{k-1\times k+1}$ is identically zero. Assertion (9) of Observation 5.6 shows that the top $k-2$ rows of B_k times t is equal to zero. We have shown that $A_\ell C = 0$. Again the ideal $I_2(C)$ has grade two since

$$\begin{vmatrix} m_k & 0 \\ 0 & M_1 \end{vmatrix} = T_1^{\ell-2} \quad \text{and} \quad \begin{vmatrix} m_1 & -M_{k-1} \\ -m_1 & M_{k+1} \end{vmatrix} \equiv T_2^{\ell-2} \quad \mathrm{mod}\ (T_1).$$

The complex (5.7.2) is also exact by the Buchsbaum-Eisenbud criteria. $\square$

We next resolve coker $\mathfrak{A}_\ell$ for the matrices $\mathfrak{A}_\ell$ of Definition 5.5.

Example 5.8. When ℓ is 5 or 6, the syzygy module for the matrices $\mathfrak{A}_5$ and $\mathfrak{A}_6$ are generated by the columns of $\mathfrak{C}_5$ and $\mathfrak{C}_6$, respectively, for

$$\mathfrak{C}_5 = \begin{bmatrix} 0 & T_2^2 \\ T_2 & 0 \\ 0 & -T_1T_2 \\ -T_1 & 0 \\ 0 & T_1^2 \end{bmatrix} \quad \text{and} \quad \mathfrak{C}_6 = \begin{bmatrix} 0 & T_2^2 \\ T_2^2 & 0 \\ 0 & -T_1T_2 \\ -T_1T_2 & 0 \\ 0 & T_1^2 \\ T_1^2 & 0 \end{bmatrix}.$$

Indeed, the rows and columns of $\mathfrak{A}_5$ and $\mathfrak{A}_6$ may be rearranged to convert these matrices into the block matrices

$$\left[\begin{array}{cc|ccc} T_1 & T_2 & 0 & 0 & 0 \\ \hline 0 & 0 & T_1 & T_2 & 0 \\ 0 & 0 & 0 & T_1 & T_2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccc|ccc} T_1 & T_2 & 0 & 0 & 0 & 0 \\ 0 & T_1 & T_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & T_1 & T_2 & 0 \\ 0 & 0 & 0 & 0 & T_1 & T_2 \end{array}\right],$$

respectively. The syzygies of the constituent pieces are well understood.

In Lemma 5.10 we prove that pattern established in Example 5.8 holds for all ℓ with $3 \le \ell$. We find it convenient to let V be a free module of rank two with basis s,t over the polynomial ring $U = k[T_1,T_2]$. The $(\ell-2)\times\ell$ matrix $\mathfrak{A}_\ell$ represents the composition

$$\text{(5.8.1)} \qquad \operatorname{Sym}_{\ell-1}V \to \operatorname{Sym}_{\ell+1}V \to \frac{\operatorname{Sym}_{\ell+1}V}{(t^{\ell+1},st^\ell,s^\ell t,s^{\ell+1})},$$

where the first map is multiplication by $\xi = T_2t^2+T_1s^2$ and the second map is the natural quotient map. The basis for $\operatorname{Sym}_{\ell-1}V$ is $t^{\ell-1},st^{\ell-2},\dots,s^{\ell-2}t,s^{\ell-1}$ and the basis for $\frac{\operatorname{Sym}_{\ell+1}V}{(t^{\ell+1},st^\ell,s^\ell t,s^{\ell+1})}$ is $s^2t^{\ell-1},s^3t^{\ell-2},\dots,s^{\ell-1}t^2$. For each positive integer α, let

$$\text{(5.8.2)} \qquad \kappa_\alpha = \sum_{a+b=\alpha}(T_2t^2)^a(-T_1s^2)^b \in \operatorname{Sym}_{2\alpha}V.$$

Notice that the columns of $\mathfrak{C}_5$ are $st\kappa_1$ and κ_2 in Sym_4V and the columns of $\mathfrak{C}_6$ are $s\kappa_2$ and $t\kappa_2$ in Sym_5V.

Lemma 5.9. *If $\ell \ge 3$ is an integer, then the kernel of the composition of* (5.8.1) *is generated by*

$$\begin{cases} st\kappa_{\frac{\ell-3}{2}} \text{ and } \kappa_{\frac{\ell-1}{2}} \text{ in } \operatorname{Sym}_{\ell-1}V & \text{if } \ell \text{ is odd} \\ s\kappa_{\frac{\ell-2}{2}} \text{ and } t\kappa_{\frac{\ell-2}{2}} \text{ in } \operatorname{Sym}_{\ell-1}V & \text{if } \ell \text{ is even.} \end{cases}$$

Proof. Observe that

$$\xi\cdot\kappa_\alpha = (T_2t^2+T_1s^2)\sum_{a+b=\alpha}(T_2t^2)^a(-T_1s^2)^b = (T_2t^2)^{\alpha+1}+(-T_1s^2)^{\alpha+1} \in \operatorname{Sym}_{2\alpha+2}V.$$

It is now clear that

$$0 \to U(-(\tfrac{\ell-1}{2}))\oplus U(-(\tfrac{\ell+1}{2})) \xrightarrow{\left[st\kappa_{\frac{\ell-3}{2}} \quad \kappa_{\frac{\ell-1}{2}}\right]} \operatorname{Sym}_{\ell-1}V(-1) \xrightarrow{(5.8.1)} \frac{\operatorname{Sym}_{\ell+1}V}{(t^{\ell+1},st^\ell,s^\ell t,s^{\ell+1})}$$

and

$$0 \to U(-(\tfrac{\ell}{2}))^2 \xrightarrow{\begin{bmatrix} s\kappa_{\frac{\ell-2}{2}} & t\kappa_{\frac{\ell-2}{2}} \end{bmatrix}} \operatorname{Sym}_{\ell-1}V(-1) \xrightarrow{(5.8.1)} \frac{\operatorname{Sym}_{\ell+1}V}{(t^{\ell+1}, st^{\ell}, s^{\ell}t, s^{\ell+1})}$$

are complexes, when ℓ is odd or even, respectively. Apply the Buchsbaum-Eisenbud criteria to see that these complexes are exact: the determinant of the top two rows is a plus or minus a power of T_2 and the determinant of the bottom two rows is plus or minus a power of T_1. $\square$

Lemma 5.10. *Adopt the notation of Definition* (5.5).

(1) *If ℓ is the even integer $2k$ and*

$$\mathfrak{C}_\ell = \begin{bmatrix} 0 & T_2^{k-1} \\ T_2^{k-1} & 0 \\ 0 & (-T_1)T_2^{k-2} \\ (-T_1)T_2^{k-2} & 0 \\ 0 & (-T_1)^2 T_2^{k-3} \\ \vdots & \vdots \\ 0 & (-T_1)^{k-1} \\ (-T_1)^{k-1} & 0 \end{bmatrix},$$

then

$$0 \to U(-k)^2 \xrightarrow{\mathfrak{C}_\ell} U(-1)^\ell \xrightarrow{\mathfrak{A}_\ell} U^{\ell-2}$$

is exact.

(2) *If ℓ is the even integer $2k+1$ and*

$$\mathfrak{C}_\ell = \begin{bmatrix} 0 & T_2^{k} \\ T_2^{k-1} & 0 \\ 0 & (-T_1)T_2^{k-1} \\ (-T_1)T_2^{k-2} & 0 \\ 0 & (-T_1)^2 T_2^{k-2} \\ \vdots & \vdots \\ (-T_1)^{k-1} & 0 \\ 0 & (-T_1)^{k} \end{bmatrix},$$

then

$$0 \to U(-k) \oplus U(-k-1) \xrightarrow{\mathfrak{C}_\ell} U(-1)^\ell \xrightarrow{\mathfrak{A}_\ell} U^{\ell-2}$$

is exact.

Proof. The present Lemma is a re-statement of Lemma 5.9. $\square$

We are now ready to prove the main result of this section. Adopt Data 5.1 with k a field closed under the taking of square root. We know from Corollary 3.10 – see also Tables 5.1.1 and 5.1.2 – that there exist bi-homogeneous elements $g_{(i,j)} \in \mathcal{A}_{(i,j)}$ such that the minimal generating set of the

B-module $\mathcal{A}$ is given by

(5.10.1)

$$\{g_{(0,d_1+d_2)}\} \cup \left\{ g_{\left(i,\frac{d_2+2-i}{2}\right)} \;\middle|\; \begin{array}{l} 1 \le i \le d_2-2 \\ \text{and } d_2-i \text{ is even} \end{array} \right\} \cup \begin{cases} \left\{ g_{\left(1,\frac{d_2+3}{2}\right)} \right\} & \text{if } d_2 \text{ is odd} \\ \left\{ g_{\left(1,\frac{d_2+2}{2}\right)}, g'_{\left(1,\frac{d_2+2}{2}\right)} \right\} & \text{if } d_2 \text{ is even.} \end{cases}$$

The element $g_{(0,d_1+d_2)}$ is the resultant of g_1 and g_2, when these polynomials are viewed as homogeneous forms in $S[x,y]$ of degree d_1 and d_2, respectively. In Theorem 5.11 we record explicit formula for the rest of the $g_{(i,j)}$ from (5.10.1). According to Observation 5.3, we may assume that the first column of φ has one of two forms. The linear form $c_{w,2}$ of S is defined in (2.6.1) by the equation

$$g_2 = \sum_{w=0}^{d_2} c_{w,2} x^w y^{d_2-w}.$$

Theorem 5.11. *Adopt Data 5.1 with k a field closed under the taking of square root. We give explicit formula for the elements of* (5.10.1)*. The element $g_{i,j}$ or $g'_{i,j}$ is in $\mathcal{A}_{i,j}$ and the set $\{g_{i,j}, g'_{i,j}\}$ of* (5.10.1) *is a bi-homogeneous minimal generating set for the B-module $\mathcal{A}$.*

(1) *Assume that the first column of φ is $[x^2+y^2, xy, 0]^{\mathrm{T}}$.*

(a) *If $1 \le i \le d_2-4$, and d_2-i is even, then*

$$g_{\left(i,\frac{d_2+2-i}{2}\right)} = \begin{cases} \sum\limits_{w=0}^{i} T_2 c_{w,2} x^w y^{i-w} m_1 + \sum\limits_{w=1}^{i} T_1 c_{w-1,2} x^w y^{i-w} m_1 + \sum\limits_{w=0}^{i} T_1 c_{w,2} x^w y^{i-w} m_2 \\ +T_1 x^i \left(\sum\limits_{\beta=2}^{\frac{d_2-i}{2}} c_{d_2+2-\beta,2} m_\beta - \sum\limits_{\beta=1}^{\frac{d_2-i}{2}} c_{\beta+i,2} m_\beta \right) \\ +(T_2 x^i + T_1 x^{i-1} y) \sum\limits_{\beta=1}^{\frac{d_2-i}{2}} (c_{d_2+1-\beta,2} - c_{\beta+i-1,2}) m_\beta, \end{cases}$$

where $m_1, \dots, m_{\frac{d_2-i}{2}}$ are the signed maximal minors of the matrix $B_{\frac{d_2-i}{2}}$ of (5.5.1)*.*

(a′) *If $i = d_2-2$ and d_2-2 is even, then*

$$g_{\left(i,\frac{d_2+2-i}{2}\right)} = \begin{cases} \sum\limits_{w=0}^{d_2-3} T_2 c_{w,2} x^w y^{d_2-2-w} + \sum\limits_{w=1}^{d_2-2} T_1 c_{w-1,2} x^w y^{d_2-2-w} - T_1 c_{d_2-1,2} x^{d_2-2} \\ -T_1 c_{d_2-2,2} x^{d_2-3} y + T_2 c_{d_2,2} x^{d_2-2} + T_1 c_{d_2,2} x^{d_2-3} y. \end{cases}$$

(b) *If $i = 1$ and d_2 is odd, then*

$$g_{\left(1,\frac{d_2+3}{2}\right)} = \begin{cases} +\underline{\chi}(d_2=3)(T_1 c_{0,2} y + T_1 c_{1,2} x) M_1 \\ -\underline{\chi}(5 \le d_2)\,(T_2 c_{0,2} y + T_2 c_{1,2} x + T_1 c_{0,2} x)\, M_{\frac{d_2-3}{2}} \\ -\underline{\chi}(7 \le d_2)(T_1 c_{0,2} y + T_1 c_{1,2} x) M_{\frac{d_2-5}{2}} - \sum\limits_{\beta=1}^{\frac{d_2-1}{2}} T_1 c_{\frac{d_2+1}{2}+\beta,2} x M_\beta \\ + \sum\limits_{\beta=1}^{\frac{d_2-3}{2}} (T_1 c_{\frac{d_2+1}{2}-\beta,2} x + T_2 c_{\frac{d_2-1}{2}-\beta,2} x + T_1 c_{\frac{d_2-1}{2}-\beta,2} y) M_\beta \\ - \sum\limits_{\beta=1}^{\frac{d_2+1}{2}} (T_2 c_{\frac{d_2-1}{2}+\beta,2} x + T_1 c_{\frac{d_2-1}{2}+\beta,2} y) M_\beta, \end{cases}$$

where $M_1,\dots,M_{\frac{d_2+1}{2}}$ are the signed maximal order minors of the matrix $B^{\dagger}_{\frac{d_2+1}{2}}$, and the matrix $B_{\frac{d_2+1}{2}}$ and the operation "†" are defined at (5.5.1) *and* (5.5.2)*, respectively.*

(c) *If $i=1$ and d_2 is even, then*

$$g_{\left(1,\frac{d_2+2}{2}\right)}=\begin{cases}(T_2c_{0,2}y+T_2c_{1,2}x+T_1c_{0,2}x)m_1+(T_1c_{0,2}y+T_1c_{1,2}x)m_2\\ -\sum\limits_{\beta=1}^{\frac{d_2}{2}}(T_1c_{1+\beta,2}x+T_2c_{\beta,2}x+T_1c_{\beta,2}y)m_\beta\\ +\sum\limits_{\beta=3}^{\frac{d_2}{2}}T_1c_{d_2+3-\beta,2}xm_\beta+\sum\limits_{\beta=2}^{\frac{d_2}{2}}(T_2c_{d_2+2-\beta,2}x+T_1c_{d_2+2-\beta,2}y)m_\beta\end{cases}$$

and

$$g'_{\left(1,\frac{d_2+2}{2}\right)}=\begin{cases}-(T_2c_{0,2}y+T_2c_{1,2}x+T_1c_{0,2}x)\,m_2-\underline{\chi}(6\le d_2)(T_1c_{0,2}y+T_1c_{1,2}x)m_3\\ +\sum\limits_{\beta=2}^{\frac{d_2}{2}}(T_1c_{\beta,2}x+T_2c_{\beta-1,2}x+T_1c_{\beta-1,2}y-T_1c_{d_2-\beta+2,2}x)m_\beta\\ -\sum\limits_{\beta=1}^{\frac{d_2}{2}}(T_2c_{1+d_2-\beta,2}x+T_1c_{1+d_2-\beta,2}y)m_\beta,\end{cases}$$

where $m_1,\dots,m_{d_2}$ are the signed maximal order minors of the matrix $B_{\frac{d_2}{2}}$ of (5.5.1).

(2) *Assume that the first column of φ is $[y^2,x^2,0]^{\mathrm{T}}$.*

(a) *If $1\le i\le d_2-2$, and d_2-i is even, then*

$$g_{\left(i,\frac{d_2+2-i}{2}\right)}=\sum_{w=0}^{i}c_{w,2}x^wy^{i-w}T_2^{\frac{d_2-i}{2}}+\sum_{\lambda=1}^{\frac{d_2-i}{2}}[c_{i+2\lambda-1,2}y+c_{i+2\lambda,2}x]x^{i-1}(-T_1)^{\lambda}T_2^{\frac{d_2-i}{2}-\lambda}.$$

(b) *If $i=1$ and d_2 is odd, then*

$$g_{\left(1,\frac{d_2+3}{2}\right)}=\sum_{\lambda=1}^{\frac{d_2+1}{2}}c_{2\lambda-1,2}y(-T_1)^{\lambda}T_2^{\frac{d_2+1}{2}-\lambda}+\sum_{\lambda=0}^{\frac{d_2-1}{2}}c_{2\lambda,2}x(-T_1)^{\lambda}T_2^{\frac{d_2+1}{2}-\lambda}.$$

(c) *If $i=1$ and d_2 is even, then*

$$\begin{aligned}g_{\left(1,\frac{d_2+2}{2}\right)}&=\sum_{\lambda=1}^{\frac{d_2}{2}}c_{2\lambda-1,2}y(-T_1)^{\lambda}T_2^{\frac{d_2}{2}-\lambda}+\sum_{\lambda=0}^{\frac{d_2}{2}}c_{2\lambda,2}x(-T_1)^{\lambda}T_2^{\frac{d_2}{2}-\lambda}\quad\textit{and}\\ g'_{\left(1,\frac{d_2+2}{2}\right)}&=\sum_{\lambda=0}^{\frac{d_2}{2}}c_{2\lambda,2}y(-T_1)^{\lambda}T_2^{\frac{d_2}{2}-\lambda}+\sum_{\lambda=0}^{\frac{d_2-2}{2}}c_{1+2\lambda,2}x(-T_1)^{\lambda}T_2^{\frac{d_2}{2}-\lambda}.\end{aligned}$$

Proof. We first prove (1.a) and (1.a′). Apply the technique of Corollary 4.4. The matrix ${\Upsilon_{d_2-i-1,1}}^{\mathrm{T}}$ of (4.4.1) is given in (5.4.1) and this matrix is called A_ℓ, with $\ell=d_2-i+1$, in Definition 5.5. The kernel of A_ℓ is calculated in part 2 of Lemma 5.7 because the index $\ell=d_2-i+1$ is odd. This kernel is free of rank two. The homogeneous minimal generating set for kernel of A_ℓ has one generator in each of two consecutive degrees. Let $\chi=[\chi_0,\dots,\chi_{\ell-1}]^{\mathrm{T}}$ be the generator with the smaller degree. The technique of Corollary 4.4 then gives

$$g_{\left(i,\frac{d_2+2-i}{2}\right)}=\sum_{\beta=0}^{\ell-1}q_{\beta,d_2-i-\beta}\chi_\beta. \tag{5.11.1}$$

An explicit formula for $q_{\beta,d_2-i-\beta}$ is given in Corollary 4.6 and

$$\chi = \begin{cases} [m_1,\dots,m_k,0,-m_k,\dots,-m_1]^{\mathrm{T}} & \text{if } 4 \le d_2-i \\ [1,0,-1]^{\mathrm{T}} & \text{if } 2 = d_2-i \end{cases} \tag{5.11.2}$$

is given in Lemma 5.7, where $k = \frac{\ell-1}{2} = \frac{d_2-i}{2}$, B_k is the matrix described in (5.5.1), and m_i is the i^{th} signed maximal order minor of B_k.

If $i = d_2-2$, then (5.11.1) yields $g_{\left(i,\frac{d_2+2-i}{2}\right)} = q_{0,2} - q_{2,0}$, with

$$q_{0,2} = \begin{cases} \sum\limits_{w=0}^{d_2-3} c_{1,1}c_{w,2}x^w y^{d_2-2-w} + \sum\limits_{w=1}^{d_2-2} c_{2,1}c_{w-1,2}x^w y^{d_2-2-w} \\ -c_{0,1}c_{d_2-1,2}x^{d_2-2} - c_{0,1}c_{d_2-2,2}x^{d_2-3}y \end{cases}$$

and $q_{2,0} = -c_{1,1}c_{d_2,2}x^{d_2-2} - c_{0,1}c_{d_2,2}x^{d_2-3}y$. The polynomial $g_1 = c_{0,1}y^2 + c_{1,1}xy + c_{2,1}x^2$ is equal to $T_1(x^2+y^2) + T_2xy$; so

$$c_{0,1} = T_1, \quad c_{1,1} = T_2, \quad \text{and} \quad c_{2,1} = T_1, \tag{5.11.3}$$

and the computation of (1.a$'$) is complete.

If $4 \le d_2-i$, then (5.11.2) gives

$$\chi_\beta = \begin{cases} m_{\beta+1} & \text{if } 0 \le \beta \le \frac{d_2-i}{2}-1 \\ 0 & \text{if } \beta = \frac{d_2-i}{2} \\ -m_{d_2-i+1-\beta} & \text{if } \frac{d_2-i}{2}+1 \le \beta \le d_2-i; \end{cases}$$

and therefore, $g_{\left(i,\frac{d_2+2-i}{2}\right)} = \theta_1 + \theta_2$, with

$$\theta_1 = \sum_{\beta=0}^{\frac{d_2-i}{2}-1} q_{\beta,d_2-i-\beta}m_{\beta+1} \quad \text{and} \quad \theta_2 = -\sum_{\beta=\frac{d_2-i}{2}+1}^{d_2-i} q_{\beta,d_2-i-\beta}m_{d_2-i+1-\beta}.$$

Use Corollary 4.6 to calculate

$$\theta_1 = \begin{cases} \sum\limits_{w=0}^{i} c_{1,1}c_{w,2}x^w y^{i-w}m_1 + \sum\limits_{w=1}^{i} c_{2,1}c_{w-1,2}x^w y^{i-w}m_1 + \sum\limits_{w=0}^{i} c_{2,1}c_{w,2}x^w y^{i-w}m_2 \\ -\sum\limits_{\beta=1}^{\frac{d_2-i}{2}} c_{0,1}c_{i+\beta,2}x^i m_\beta - \sum\limits_{\beta=1}^{\frac{d_2-i}{2}} c_{1,1}c_{i+\beta-1,2}x^i m_\beta - \sum\limits_{\beta=1}^{\frac{d_2-i}{2}} c_{0,1}c_{i+\beta-1,2}x^{i-1}ym_\beta \end{cases}$$

and

$$\theta_2 = \sum_{\beta=2}^{\frac{d_2-i}{2}} c_{0,1}c_{d_2+2-\beta,2}x^i m_\beta + \sum_{\beta=1}^{\frac{d_2-i}{2}} c_{1,1}c_{d_2+1-\beta,2}x^i m_\beta + \sum_{\beta=1}^{\frac{d_2-i}{2}} c_{0,1}c_{d_2+1-\beta,2}x^{i-1}ym_\beta.$$

Use (5.11.3) to complete the proof of (1.a).

To prove (1.b), we consider the minimal syzygy of degree k

$$\chi = \begin{bmatrix} -M_{k-1} & \dots & -M_1 & 0 & M_1 & \dots & M_{k+1} \end{bmatrix}^{\mathrm{T}}$$

of the matrix A_{d_2} , where $k = \frac{d_2-1}{2}$, as given in part (2) of Lemma 5.7. This formulation makes sense and gives

$$\chi = \begin{bmatrix} 0 & T_2 & -T_1 \end{bmatrix}^{\mathrm{T}}$$

when $d_2=3$. In other words, $\chi=[\chi_0,\dots,\chi_{d_2-1}]^{\mathrm{T}}$, with

$$\chi_\beta=\begin{cases}-M_{\frac{d_2-3}{2}-\beta} & \text{if } 0\le\beta\le\frac{d_2-5}{2}\\ 0 & \text{if } \beta=\frac{d_2-3}{2}\\ M_{\beta+1-\frac{d_2-1}{2}} & \text{if } \frac{d_2-1}{2}\le\beta\le d_2-1.\end{cases}$$

The techniques of Corollary 4.4 give

$$g_{\left(1,\frac{d_2+3}{2}\right)}=\sum_{\beta=0}^{\ell-1}q_{\beta,d_2-i-\beta}\chi_\beta=-\sum_{\beta=0}^{\frac{d_2-1}{2}-2}q_{\beta,d_2-i-\beta}M_{\frac{d_2-3}{2}-\beta}+\sum_{\beta=\frac{d_2-1}{2}}^{d_2-1}q_{\beta,d_2-i-\beta}M_{\beta-\frac{d_2-1}{2}+1}.$$

We apply Corollary 4.6 and (5.11.3) as we simplify this expression.

The computation of (1.c) proceeds in the same manner. One begins with the relations

$\chi=\begin{bmatrix}m_1 & \dots & m_{\frac{d_2}{2}} & 0 & -m_{\frac{d_2}{2}} & \dots & -m_2\end{bmatrix}^{\mathrm{T}}$ and $\chi'=\begin{bmatrix}-m_2 & \dots & -m_{\frac{d_2}{2}} & 0 & m_{\frac{d_2}{2}} & \dots & m_1\end{bmatrix}^{\mathrm{T}}$

on the matrix A_{d_2}, as given in part (1) of Lemma 5.7, where the matrix $B_{\frac{d_2}{2}}$ is given in (5.5.1), and $m_1,\dots,m_{d_2}$ are the maximal order minors of $B_{\frac{d_2}{2}}$. The relations χ and χ' are used to produce $g_{\left(1,\frac{d_2+2}{2}\right)}$ and $g'_{\left(1,\frac{d_2+2}{2}\right)}$, respectively.

Now we prove (2.a). We again use (5.11.1). This time the relation

$$\chi=\begin{bmatrix}0\\ T_2^{\alpha}\\ 0\\ (-T_1)T_2^{\alpha-1}\\ \vdots\\ (-T_1)^{\alpha}\\ 0\end{bmatrix},$$

with $\alpha=\frac{d_2-i-2}{2}$, is read from Lemma 5.10. We see that

$$\chi_\beta=\begin{cases}0 & \text{if } \beta \text{ is even}\\ (-T_1)^{\lambda}T_2^{\frac{d_2-i-2}{2}-\lambda} & \text{if } \beta=2\lambda+1 \text{ and } 0\le\lambda\le\frac{d_2-i-2}{2}.\end{cases}$$

Use (5.11.1) and Corollary 4.6 to see that $g_{\left(i,\frac{d_2+2-i}{2}\right)}$ is equal to

$$\begin{aligned}
&= \sum_{\lambda=0}^{\frac{d_2-i}{2}-1}q_{2\lambda+1,d_2-i-(2\lambda+1)}(-T_1)^{\lambda}T_2^{\frac{d_2-i}{2}-\lambda-1}\\
&= \left\{\begin{array}{l}\displaystyle\sum_{w=0}^{i}c_{2,1}c_{w,2}x^wy^{i-w}T_2^{\frac{d_2-i-2}{2}}\\ \displaystyle+\sum_{\lambda=0}^{\frac{d_2-i}{2}-1}\left(-c_{0,1}c_{i+2\lambda+2,2}x^i-c_{1,1}c_{i+2\lambda+1,2}x^i-c_{0,1}c_{i+2\lambda+1,2}x^{i-1}y\right)(-T_1)^{\lambda}T_2^{\frac{d_2-i}{2}-\lambda-1}.\end{array}\right.
\end{aligned}$$

The polynomial $g_1=c_{0,1}y^2+c_{1,1}xy+c_{2,1}x^2$ is equal to $T_1y^2+T_2x^2$; so

(5.11.4) $$c_{0,1}=T_1,\quad c_{1,1}=0,\quad \text{and}\quad c_{2,1}=T_2,$$

and the computation of (2.a) is complete.

The computations of (2.b) and (2.c) proceed in the same manner. One uses

$$\chi=\begin{bmatrix} T_2^{\frac{d_2-1}{2}}\\ 0\\ \vdots\\ 0\\ (-T_1)^{\frac{d_2-1}{2}}\end{bmatrix},\quad \chi=\begin{bmatrix} T_2^{\frac{d_2}{2}-1}\\ 0\\ \vdots\\ 0\\ (-T_1)^{\frac{d_2}{2}-1}\\ 0\end{bmatrix},\quad\text{and}\quad \chi=\begin{bmatrix} 0\\ T_2^{\frac{d_2}{2}-1}\\ 0\\ \vdots\\ 0\\ (-T_1)^{\frac{d_2}{2}-1}\end{bmatrix}$$

to compute $g_{\left(1,\frac{d_2+3}{2}\right)}$ (when d_2 is odd), and $g_{\left(1,\frac{d_2+2}{2}\right)}$ and $g'_{\left(1,\frac{d_2+2}{2}\right)}$ (when d_2 is even), respectively. □

6. THE CASE OF $d_1 = d_2$.

The S-module structure of $\mathcal{A}_{\ge d_2-1}$ is completely described in Corollary 2.16 for all choices of $d_1 \le d_2$ in Data 2.1. If $d_1 < d_2$ and the first column of φ contains a generalized zero, then the S-module structure of $\mathcal{A}_{\ge d_1-1}$ is completely described in Theorem 3.3; see also Table 3.5. In Theorem 6.2 we assume that $d_1 = d_2$ and we describe $\mathcal{A}_{d_1-2}$; hence, in this case, the S-module structure of $\mathcal{A}_{\ge d_1-2}$ is completely described by combining Theorem 6.2 and Corollary 2.16. The geometric significance of Theorem 6.2 is explained in the Remarks 6.3. A preliminary version of Theorem 6.2 initiated the investigation that culminated in [6]. Most of the calculations that are used in the proof of Theorem 6.2 have already been incorporated in [6]. The geometrical applications of these calculations are emphasized in [6]. In the present work we focus on the application of these calculations to Rees algebras.

Data 6.1. Adopt Data 2.1 with $d_1 = d_2$. Let

$$C=\begin{bmatrix} c_{0,1} & c_{0,2}\\ \vdots & \vdots\\ c_{d_1,1} & c_{d_1,2}\end{bmatrix}$$

be the matrix $[\Upsilon_{1,1}|\Upsilon_{1,2}]$ of Definition 2.6. Notice that

(6.1.1) $$\begin{bmatrix} T_1 & T_2 & T_3\end{bmatrix}\varphi=\begin{bmatrix} y^{d_1} & xy^{d_1-1} & \cdots & x^{d_1}\end{bmatrix}C.$$

Theorem 6.2. *If Data* 6.1 *is adopted, then the following statements hold.*

(1) *There are eight possible values for the pair of integers* $(\mu(I_1(\varphi)),\mu(I_2(C)))$. *Indeed, the following chart gives the possible values for* $(\mu(I_1(\varphi)),\mu(I_2(C)))$ *as a function of* d_1:

d_1	*possible values for* $(\mu(I_1(\varphi)),\mu(I_2(C)))$
5 *or more*	$(6,6)$, $(5,6)$, $(5,5)$, $(4,6)$, $(4,5)$, $(4,4)$, $(3,3)$, *or* $(2,1)$
4	$(5,6)$, $(5,5)$, $(4,6)$, $(4,5)$, $(4,4)$, $(3,3)$, *or* $(2,1)$
3	$(4,6)$, $(4,5)$, $(4,4)$, $(3,3)$, *or* $(2,1)$
2	$(3,3)$, *or* $(2,1)$
1	$(2,1)$.

(2) *The S-module $\mathcal{A}_{d_1-2}$ is resolved by*

$$\begin{array}{ccccll}
0 & \to & S(-4)^2 & \to & S(-2)^{d_1-5}\oplus S(-3)^6 & \textit{if } (\mu(I_1(\varphi)),\mu(I_2(C)))=(6,6)\\
0 & \to & S(-5)^1 & \to & S(-2)^{d_1-4}\oplus S(-3)^3\oplus S(-4)^1 & \textit{if } (\mu(I_1(\varphi)),\mu(I_2(C)))=(5,6)\\
0 & \to & S(-4)^1 & \to & S(-2)^{d_1-4}\oplus S(-3)^4 & \textit{if } (\mu(I_1(\varphi)),\mu(I_2(C)))=(5,5)\\
0 & \to & S(-5)^2 & \to & S(-2)^{d_1-3}\qquad\quad\oplus S(-4)^4 & \textit{if } (\mu(I_1(\varphi)),\mu(I_2(C)))=(4,6)\\
0 & \to & S(-5)^1 & \to & S(-2)^{d_1-3}\oplus S(-3)\oplus S(-4)^2 & \textit{if } (\mu(I_1(\varphi)),\mu(I_2(C)))=(4,5)\\
 & & 0 & \to & S(-2)^{d_1-3}\oplus S(-3)^2 & \textit{if } (\mu(I_1(\varphi)),\mu(I_2(C)))=(4,4)\\
 & & 0 & \to & S(-2)^{d_1-2}\qquad\quad\oplus S(-4)^1 & \textit{if } (\mu(I_1(\varphi)),\mu(I_2(C)))=(3,3)\\
 & & 0 & \to & S(-2)^{d_1-1} & \textit{if } (\mu(I_1(\varphi)),\mu(I_2(C)))=(2,1).
\end{array}$$

(3) *The S-module $\mathcal{A}_{d_1-2}$ is free if and only if $\mu(I_2(C))\le 4$.*

Proof. We know from Theorem 2.7, part (1), with $i=d_1-2$, that

$$0\to\mathcal{A}_{d_1-2}\to S(-2)^{d_1+1}\xrightarrow{C^{\mathrm{T}}}S(-1)^2$$

is an exact sequence of homogeneous S-module homomorphisms; thus, $\mathcal{A}_{d_1-2}\simeq\ker C^{\mathrm{T}}$. We apply the results of Section 4 in [6].

The matrix φ is an element of the space of matrices "BalH_d" from Definition 4.3 in [6]. The group $G=\mathrm{GL}_3(k)\times\mathrm{GL}_2(k)$ acts on BalH_d; and, in Theorem 4.9 of [6], BalH_d is decomposed into 11 disjoint orbits under the action of G: $\mathrm{BalH}_d=\bigcup_{\#\in\mathrm{ECP}}\mathrm{DO}_{\#}^{\mathrm{Bal}}$. These orbits are parameterized by a poset called the Extended Configuartion Poset (ECP). The value of $(\mu(I_1(\varphi)),\mu(I_2(C)))$, as a function of # with $\varphi\in\mathrm{DO}_{\#}^{\mathrm{Bal}}$, is given, in Part (2) of Lemma 4.10 in [6], to be

$$\begin{array}{llll}
(\mu(I_1(\varphi)),\mu(I_2(C)))=(6,6) & \iff & \#=(\emptyset,\mu_6) & \\
(\mu(I_1(\varphi)),\mu(I_2(C)))=(5,6) & \iff & \#=(\emptyset,\mu_5) & \\
(\mu(I_1(\varphi)),\mu(I_2(C)))=(5,5) & \iff & \#=(c,\mu_5) & \\
(\mu(I_1(\varphi)),\mu(I_2(C)))=(4,6) & \iff & \#=(\emptyset,\mu_4) & \\
(\mu(I_1(\varphi)),\mu(I_2(C)))=(4,5) & \iff & \#=(c,\mu_4) & \\
(\mu(I_1(\varphi)),\mu(I_2(C)))=(4,4) & \iff & \#=(c,c), & \text{or } (c:c)\\
(\mu(I_1(\varphi)),\mu(I_2(C)))=(3,3) & \iff & \#=(c,c,c), & \text{or } (c:c,c),\quad \text{or } (c:c:c)\\
(\mu(I_1(\varphi)),\mu(I_2(C)))=(2,1) & \iff & \#=\mu_2. &
\end{array}$$

We notice that when d_1 is small, then it is not possible for $(\mu(I_1(\varphi)),\mu(I_2(C)))$ to take on all of the values listed so far. Indeed, the entries of φ are elements of the vector space of homogeneous forms of degree d_1 in 2 variables; consequently, $I_1(\varphi)\le d_1+1$. On the other hand, this is the only constraint that a small value for d_1 imposes on the pair $(\mu(I_1(\varphi)),\mu(I_2(C)))$, as is shown in Proposition 4.21 of [6]. This completes the proof of (1).

Now we prove (2). For each # in ECP, there is a canonical matrix $C_{\#}$ with the property that if $\varphi'\in\mathrm{DO}_{\#}^{\mathrm{Bal}}$ and C' is the partner of φ' in the sense of (6.1.1), then $C_{\#}$ may be obtained from C' by using elementary row and column operations and the suppression of zero rows; furthermore, $I_\ell(C')=I_\ell(C_{\#})$, and if $X_{\#}=\left[\begin{smallmatrix}C_{\#}\\0\end{smallmatrix}\right]$ is the matrix with the same number of rows as C', then $\ker C'^{\mathrm{T}}\simeq\ker X_{\#}^{\mathrm{T}}$. We

now record the matrices $C'_\#$ as given in [6, Lemma 4.10, part (1)]:

$$\begin{array}{llll}
C_{(\emptyset,\mu_6)} = \begin{bmatrix} T_1 & 0\\ T_2 & 0\\ T_3 & 0\\ 0 & T_1\\ 0 & T_2\\ 0 & T_3\end{bmatrix}, & C_{(\emptyset,\mu_5)} = \begin{bmatrix} T_1 & T_3\\ T_2 & 0\\ T_3 & 0\\ 0 & T_1\\ 0 & T_2\end{bmatrix}, & C_{(c,\mu_5)} = \begin{bmatrix} T_1 & 0\\ T_2 & 0\\ 0 & T_1\\ 0 & T_2\\ 0 & T_3\end{bmatrix}, & C_{(\emptyset,\mu_4)} = \begin{bmatrix} T_1 & 0\\ T_2 & T_1\\ T_3 & T_2\\ 0 & T_3\end{bmatrix},\\
C_{(c,\mu_4)} = \begin{bmatrix} T_1 & T_2\\ T_2 & 0\\ 0 & T_1\\ 0 & T_3\end{bmatrix}, & C_{c,c} = \begin{bmatrix} T_1 & 0\\ 0 & T_1\\ T_2 & T_2\\ 0 & T_3\end{bmatrix}, & C_{c:c} = \begin{bmatrix} T_1 & 0\\ T_2 & T_3\\ 0 & T_1\\ 0 & T_2\end{bmatrix}, & C_{c:c:c} = \begin{bmatrix} T_1 & T_2\\ T_2 & T_3\\ 0 & T_1\end{bmatrix},\\
C_{c:c,c} = \begin{bmatrix} T_1 & 0\\ T_2 & T_3\\ 0 & T_2\end{bmatrix}, & C_{c,c,c} = \begin{bmatrix} T_1 & T_1\\ T_2 & 0\\ 0 & T_3\end{bmatrix}, & C_{\mu_2} = \begin{bmatrix} T_1 & T_2\\ T_2 & T_3\end{bmatrix}. &
\end{array}$$

One readily checks that the kernel of $X_\#$ is as claimed for each choice of #. This completes the proof of (2). Assertion (3) follows immediately from (2). $\square$

Remarks 6.3. We explain the names of the elements of ECP. Adopt Data 6.1 with k algebraically closed. Let $\mathcal{C}_\varphi$ be the curve and $\eta_\varphi : \mathbb{P}^1 \to \mathcal{C}_\varphi$ be the parameterization described in Remark 2.9.

(1) If $\varphi \in \mathrm{DO}^{\mathrm{Bal}}_{\mu_2}$, then η_φ is a birational parameterization of $\mathcal{C}_\varphi$ if and only if $d_1 = 1$; in this case, η_ϕ is non-singular.

(2) If d_1 is a prime integer, then η_φ is a birational parameterization of $\mathcal{C}_\varphi$ if and only if $\mu(I_1(\varphi))$ is at least 3; see [6, Obs. 0.11].

For the rest of our remarks, we assume that η_φ is a birational parameterization of $\mathcal{C}_\varphi$. (If one starts with a non-birational parameterization, then one can always re-parameterize in order to obtain a birational parameterization.)

(3) If $\varphi \in \mathrm{DO}^{\mathrm{Bal}}_\#$, for some $\# \in$ ECP, and η_φ is a birational parameterization of $\mathcal{C}_\varphi$, then the number of c's that appear in the name of # is equal to the number of singularities of multiplicity d_1 on, or infinitely near, $\mathcal{C}$. (In [6], the degree of $\mathcal{C}$ is $d = 2c$; in the present language, the degree of $\mathcal{C}$ is $d = 2d_1$.) In particular,
   (a) The S-module $\mathcal{A}_{d_1-2}$ is free and isomorphic to $S(-2)^{d_1-2} \oplus S(-4)^1$ if and only if there are exactly 3 singularities of multiplicity d_1 on, or infinitely near, $\mathcal{C}$.
   (b) The S-module $\mathcal{A}_{d_1-2}$ is free and isomorphic to $S(-2)^{d_1-3} \oplus S(-3)^2$ if and only if there are exactly 2 singularities of multiplicity d_1 on, or infinitely near, $\mathcal{C}$.
   (c) The S-module $\mathcal{A}_{d_1-2}$ is free if and only if there are at least 2 singularities of multiplicity d_1 on, or infinitely near, $\mathcal{C}$.

(4) If # is an element of ECP, then the punctuation describes the configuration of multiplicity d_1 singularities on $\mathcal{C}$. A colon indicates an infinitely near singularity, a comma indicates a different singularity on the curve, and $\emptyset$ indicates that there are no multiplicity d_1 singularities on $\mathcal{C}_\varphi$.

Remark 6.4. Again, we compare our results with the results of [3]. In [3] the ambient hypothesis forces d_2 to equal either d_1 or $d_1 + 1$ (see [3, Cor. 4.4]); and therefore, the hypothesis $d_1 = d_2$ of the present section agrees with one of the hypotheses of [3]. The style of answer; however, is completely different. We describe **all** of the possible bi-degrees for all possible bi-homogeneous

minimal generating sets for **exactly one** $\mathcal{A}_i$; namely, $\mathcal{A}_{d_1-2}$, and we explain how these bi-degrees reflect the configuration of singularities on the curve. On the other hand, [3] considers **exactly one** configuration of singularities and for this configuration [3] gives a **complete set** of minimal bi-homogeneous generators (notice generators rather than just degrees) for the **entire ideal** $\mathcal{A}$. The singularities of the curve of [3] can be read from [3, Thm. 3.11], together with [6, Cor. 1.9(1)]. The curve of [3] has four singularities: all of these singularities are on the curve, three of these singularities have multiplicity d_1, and the fourth singularity has multiplicity d_1-1. In the language of the present section, the singularities of the curve of [3] correspond to the element $\# = (c,c,c)$ of the Extended Configuration Poset and the Data 6.1 for the curves of [3] satisfies $(\mu(I_1(\varphi)),\mu(I_2(C))$ is equal to $(3,3)$.

7. An Application: Sextic curves.

The results outlined in the previous sections suffice to provide significant information about the defining equations for $\mathcal{R}$ if $d = d_1+d_2 \le 6$, since then $d_1 \le 2$ (see Section 5) or $d_1 = d_2$ (see Section 6). We focus on the case $d = 6$, the case of a sextic curve. The following data is in effect throughout this section.

Data 7.1. Adopt Data 2.1 with $d = 6$ and k an algebraically closed field. Let $\eta : \mathbb{P}^1_k \to \mathbb{P}^2_k$ be the morphism determined by φ and $\mathcal{C}$ be the rational plane curve parameterized by η, as described in Remark 2.9. Assume that $\mathcal{C}$ has degree 6, or equivalently, that the morphism η is birational onto its image $\mathcal{C}$. Let $\mathcal{J}$ be the ideal in B which is the kernel of the composition

$$B \to \operatorname{Sym}(I) \to \mathcal{R},$$

as described in Data 2.1.

As it turns out, there is, essentially, a one-to-one correspondence between the bi-degrees of the defining equations of $\mathcal{R}$ on the one hand and the types of the singularities on or infinitely near the curve $\mathcal{C}$ on the other hand. Here one says that a singularity is infinitely near $\mathcal{C}$ if it is obtained from a singularity on $\mathcal{C}$ by a sequence of quadratic transformations. This correspondence can be justified using the results of [6]. The information in Theorem 7.2 about the bi-degrees of the defining equations of $\mathcal{R}$ is a compilation of results from many places as described below.

Theorem 7.2. *Adopt Data* 7.1. *The correspondence between the bi-degrees of the defining equations of $\mathcal{R}$ and the types of the singularities on or infinitely near the curve $\mathcal{C}$ is summarized in Table* 7.2.1. *The first column gives the possible values of d_1,d_2, namely* $1,5$ *or* $2,4$ *or* $3,3$; *the second column lists the corresponding bi-degrees of minimal generators of $\mathcal{J}$ together with the multiplicities by which they appear, suppressing the obvious bi-degrees* $(d_1,1),(d_2,1)$ *(of the equations defining* $\operatorname{Sym}(I)$*) and* $(0,6)$ *(of the implicit equation of* $\mathcal{C}$*); and the third column gives the multiplicities of the singularities on or infinitely near $\mathcal{C}$.*

d_1	d_2	equations of $\mathcal{R}$	singularities of $\mathcal{C}$
1	5	(1,5) (2,4) (3,3) (4,2)	1 of multiplicity 5 on $\mathcal{C}$
2	4	(1,3):2 (2,2)	1 of multiplicity 4 on $\mathcal{C}$ 4 double points on or near $\mathcal{C}$
		(1,4):4 (2,3):3 (3,2)	10 double points on or near $\mathcal{C}$
3	3	(1,4):4 (2,2):3	10 double points on or near $\mathcal{C}$
		(1,3) (1,4):2 (2,2):3	1 of multiplicity 3 on $\mathcal{C}$ 7 double points on or near $\mathcal{C}$
		(1,3):2 (2,2):3	2 of multiplicity 3 and 4 double points on or near $\mathcal{C}$
		(1,2) (1,4) (2,2)	3 of multiplicity 3 and 1 double point on or near $\mathcal{C}$

Table 7.2.1. The correspondence between the Rees algebra and the singularities of a parameterized plane sextic.

Remark 7.3. Notice that in Table 7.2.1, the constellation of 10 double points on or infinitely near the curve corresponds to two distinct numerical types of Rees algebras. Thus, the Rees algebra provides a finer distinction. We would like to know a geometric interpretation of this algebraic distinction.

Before proving Theorem 7.2, we recall a few of the ingredients that are used. We make repeated use of Max Noether's formula for the geometric genus of an irreducible plane curve. In the special case of a rational curve $\mathcal{C}$ of degree 6 it says that

$$10=\sum_q \binom{m_q}{2}, \tag{7.3.1}$$

where q ranges over all singularities on or infinitely near $\mathcal{C}$, and m_q is the multiplicity at q. We also make repeated use of the General Lemma of [6] (see [6, Lem. 1.7, Thm. 1.8, and Cor. 1.9], or [1, Lem. 1.1] and [2, Lems. 1.3 and 1.5 and Prop. 1.5] or [26, Thm. 3]), which we again state under the special hypotheses of this section.

Theorem 7.4. *Adopt Data* 7.1.

(1) *If p is a point on $\mathcal{C}$, then the multiplicity m_p of $\mathcal{C}$ at p satisfies either $m_p=d_2$, or else $m_p\le d_1$.*
(2) *If $d_1<d_2$, then column* 1 *of* φ *has a generalized zero if and only if $\mathcal{C}$ has a singularity of multiplicity d_2.*
(3) *If $d_1=d_2=3$ and C is the matrix of Theorem* 6.2*, then the number of distinct singular points of multiplicity* 3 *that are either on $\mathcal{C}$ or infinitely near to $\mathcal{C}$ is $6-\mu(I_2(C))$.*
(4) *The infinitely near singularities of $\mathcal{C}$ have multiplicity at most d_1.*

Remark. Item (3) in the above result is established by combining parts (1) and (4) of Theorem 3.22 of [6]. Notice that the equality $\deg\gcd I_3(A)=6-\mu(I_2(C))$ from (1) of [6, Thm. 3.22] holds even

when $\gcd I_3(A)$ is a unit; see, for example, item (2) of [6, Lemma 4.10]. Item (4) in the above result is [6, Cor. 2.3].

Proof of Theorem 7.2. We now justify the Table 7.2.1 in detail. When $(d_1,d_2)=(1,5)$, the numerical information about the Rees ring follows from [15, Prop. 4.3], [7, Thm. 2.3], or [23, Thm. 3.6]. Moreover, there is a generalized zero in the first column of φ, hence the curve $\mathcal{C}$ has a singularity of multiplicity 5, as can be seen from item (2) of Theorem 7.4. There are no further singularities or infinitely near singularities of $\mathcal{C}$ by Max Noether's formula (7.3.1).

Next assume that $(d_1,d_2)=(2,4)$ and that the first column of φ has a generalized zero. The numerical information about the Rees algebra is contained in Table 5.1.2. Item (1) of Theorem 7.4 shows that all of the singularities of $\mathcal{C}$ have multiplicity 2 or 4, and, according to (2), $\mathcal{C}$ has a singularity of multiplicity 4. All of the infinitely near singularities of $\mathcal{C}$ have multiplicity 2 by (4). The rest of the description of the singularities of $\mathcal{C}$ follows from (7.3.1).

If $(d_1,d_2)=(2,4)$ and the first column of φ does not have a generalized zero, then the numerical information about the Rees ring is given by [4, Prop. 3.2]. Since the first column of φ does not have a generalized zero there is no point of multiplicity 4 on the curve and hence all singularities and infinitely near singularities have multiplicity 2 and there are 10 of them by (7.3.1).

Finally we consider the case $(d_1,d_2)=(3,3)$. By Theorem 7.4, all of the singularities on or infinitely near $\mathcal{C}$ have multiplicity either 2 or 3; and therefore, we employ (7.3.1), once again, to see that the last column of Table 7.2.1 lists all possible configurations of singularities on or infinitely near $\mathcal{C}$. We use Theorem 6.2 to connect the configuration of singularities on or infinitely near $\mathcal{C}$ to the degrees of the defining equations of $\mathcal{R}$. Part (3) of Theorem 7.4 shows that the number of points of multiplicity 3 on or infinitely near $\mathcal{C}$ is $6-\mu(I_2(C)))$. The other statistic which is used in from Theorem 6.2 is $\mu(I_1(\varphi))$. Observe that $3\le\mu(I_1(\varphi))\le 4$. Indeed, the first inequality holds by part (1) of Remark 6.3 because the ambient hypothesis of Data 7.1 guarantees that the morphism η, which parameterizes the curve $\mathcal{C}$, is birational onto its image; and the second inequality holds because the entries of φ are cubics in two variables. We read the T-degrees of a minimal S-module generating set for $\mathcal{A}_1$ from part (2) of Theorem 6.2. As seen above, we need only look at the rows with $3\le\mu(I_1(\varphi))\le 4$. We see that if there are no multiplicity 3 singularities on or infinitely near $\mathcal{C}$, then $\mu(I_2(C))=6$ and there are 4 minimal generators of the S-module $\mathcal{A}_1$ and each of these has T-degree 4. We record these generator degrees as $(1,4):4$ since every element of $\mathcal{A}_1$ has xy-degree 1. Similarly, if there is exactly 1 multiplicity 3 singularity on or infinitely near $\mathcal{C}$, then the minimal generators of the S-module $\mathcal{A}_1$ have bi-degree $(1,3)$ and $(1,4):2$. If there are two such singularities then the generators have bi-degree $(1,3):2$ and if there are 3 such singularities then the generators have bi-degree $(1,2)$ and $(1,4)$. Corollary 2.16 shows that the S-module $\mathcal{A}_{\ge 2}$ is minimally generated by 3 elements of bi-degree $(2,2)$. Degree considerations show that in three of the cases the minimal generators of the B-module $\mathcal{J}$ have been identified. When there are 3 singularities of multiplicity 3 on or infinitely near $\mathcal{C}$, then a further calculation is necessary. As $\mathrm{Sym}(I)_{\le 2}\simeq B_{\le 2}$, multiplying the

generator of $\mathcal{A}$ of bi-degree $(1,2)$ yields two linearly independent elements of $\mathcal{A}$ of bi-degree $(2,2)$; and therefore $\dim_k[\mathcal{A}/B\mathcal{A}_{(1,2)}]_{(2,2)} = 1$. $\square$

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